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A model of trust building with anonymous re-matching

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A R T I C L E I N F O

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A B S T R A C T

This paper studies a repeated lender-borrower game with anonymous re-matching (that is, once an ongoing relationship is terminated, players are rematched with new partners and prior histories are unobservable). We propose an equilibrium refinement based on two assumptions: (a) default implies termination of the current relationship; (b) in a given relationship, a better loan-repayment history implies weakly higher continuation values for both parties. This refinement captures the idea of "justifiable punishments" in repeated games. We show that if agents are patient enough and re-matching is sufficiently likely, then the loan size is strictly increasing over time along the equilibrium path of *all* non-trivial equilibria. As such, this paper helps explain gradualism in long-term relationships, especially credit relationships.

1. Introduction

Gradualism, or "starting small", is often observed in long-term relationships. It is reflected in the increasing level of interactions over time between parties in a relationship. For example, the credit line granted by a credit card company to a specific customer normally increases over time as more and more on-time repayments are made; Antràs and Foley, (2015) document the pattern of financing terms of a U.S.-based exporter in the poultry industry, and find that the amount of trade credit granted by the exporter to its trading partners increases with the length of their relationships. More broadly, in large societies or informal credit markets of developing economies where personal history records are hard to obtain, people are usually cautious at the beginning of a new relationship, and put more at stake after satisfactory interactions.

This paper focuses on one specific driving force of gradualism: the re-matching opportunity. We study a repeated lender-borrower game with anonymous re-matching, in which a lender and a borrower interact in a society with a group of lenders and a group of borrowers. In addition to lending and repayment decisions, the lender and borrower can choose whether or not to continue their relationship at the end of each period. If a relationship is terminated, each party becomes unmatched; in the next period, an unmatched agent will be anonymously matched with a new partner with some exogenous probability. The re-matching is *anonymous* in the sense that the agents' actions in previous relationships are unobservable in the current

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relationship. This captures the idea that it is costly to acquire past history information of the other party in a new relationship. To highlight the importance of re-matching, the model abstracts away from incomplete information about agents' type, and commitment power.

The model and the question studied in this paper are reminiscent of Datta (1996), who looks at a special case of our model with immediate rematching and linear payoffs, and shows that the *value of the relationship*—defined as the discounted sum of current and future loans—is nondecreasing over time for *efficient* equilibria. My paper is substantially different in the following three aspects. First, our setup is more general by allowing for probabilistic re-matching and richer payoff structures. Second, this paper proves a result of strictly increasing *loans*, not just nondecreasing values. This is a sharper prediction because Datta's result regarding value monotonicity still permits non-monotone sequence of loans[2]; it is also easier to test because the value of a relationship is much harder to observe than the level of interactions in each period. Third, we propose and focus on a reasonable equilibrium refinement related to the notion of "justifiable punishments" (Aramendia and Wen, 2014). It addresses the question of what should happen after the players deviate to an off-equilibrium but "better" history. Since efficiency is not imposed in the refinement, our sharper prediction holds even for *inefficient* equilibria.

Specifically, we focus on equilibria in which the same strategies are used in every relationship; that is, every new relationship is just a restart of the first relationship. We call such a strategy profile a *social equilibrium*. Such equilibria are studied by Datta (1996) with a focus on the efficient ones, and by Ghosh and Ray (1996, 2016) in a setting with a simultaneous stage game and incomplete information about players' types. We propose a refinement called *orthodox social equilibria*, which are social equilibria such that: (i) an ongoing relationship is terminated by the lender on default; (ii) a better loan-repayment history is followed by weakly higher continuation values for the lender and the borrower. Our main result is that, if the discount factor and the re-matching probability are above certain thresholds, the size of loans along the equilibrium path of *any* (non-trivial) orthodox social equilibrium is strictly increasing over time.

In the definition of orthodox social equilibrium, the first restriction that the relationship is terminated on default is standard in the literature (for example, see Datta, 1996; Ghosh and Ray, 1996; Ghosh and Ray, 2016; Kranton, 1996). The second restriction is motivated by the following idea. In credit relationships, repayment is viewed as a better action than default because default directly hurts the lender. In addition, a larger loan is better than a smaller one, in the sense that given that repayment happens, a larger loan leads to higher per-period payoffs for both parties. At the beginning of a given date t,[3] if there are two histories of actions such that neither entails default and they differ only in the loan size in the last period, we may then view the history with a higher last-period loan as a result of the lender's deviation to offering an (unexpected) higher loan followed by the borrower's repayment. The lender's (and potentially the borrower's) deviation benefits both parties in this scenario, so the history with a higher last-period loan is arguably better. The second restriction in defining orthodox equilibrium says that, at any given t, the continuation values of the relationship following a better loan-repayment history should be weakly higher; otherwise, if they are strictly lower, agents can be viewed as punishing each other by using strictly lower continuation values, even though both of them have benefited from the deviation. But punishing the deviating player for taking better actions is not "morally justified" because she did not hurt anyone; moreover, it is not even "economically justified" if the punishment is costly to the non-deviating player (who is the "carrier" of the punishment).[4] The idea behind this refinement is related to the notion of "justifiable punishments" in normal-form repeated games (Aramendia and Wen, 2014) which also puts a monotone restriction on continuation values following deviations that benefit the other party.

One important implication of our refinement is that, in any orthodox social equilibrium, the lender will make the borrower's no-deviation (i.e. no-default) constraint bind at all dates. This is because if it is not binding at some date, i.e. the borrower strictly prefers repaying the loan, then the lender can increase the loan size at that date by a little without inducing default. The reason why the borrower will not default on the slightly higher loan is as follows. If the borrower repays, she will enjoy a weakly higher continuation value than before because repaying a higher loan results in a better history than the one on path[5]; since her no-deviation constraint is not binding on path, the same constraint is still satisfied at a slightly higher loan followed by a weakly higher continuation value. But if a slightly higher loan does not induce default, by offering it the lender will also enjoy a weakly higher continuation value (together with a strictly higher current payoff) because a better loan-repayment history will be achieved. So in any orthodox social equilibrium, the lender has an incentive to

[2] For example, consider the following nondecreasing value sequence: $\{375, 425, 500, 500, 500, \ldots\}$. When $\delta = 0.8$, it can be generated the following loan sequence: $\{35, 25, 100, 100, 100, \ldots\}$. This loan sequence is non-monotone.

[3] The beginning of date t is the first subperiod of t when the lender decides how much to lend. A history at that node consists of all the actions (loan sizes, repayment decisions and continuation decisions) from date 0 to $t-1$.

[4] To be more concrete, consider the following situation. Suppose that on an equilibrium path the lender should offer a loan of \$1000 in the current period, but she deviates to lending \$2000 and the borrower later repays. This leads them to an off-equilibrium history. At the end of that period, should the deviating player(s) be punished (for example, via terminating the relationship) just for the sake of the fact that some deviation has happened? More generally, should the value of a person's relationship with a bank be strictly reduced after the bank (unexpectedly) lends more and the person repays? Although subgame perfection does not offer an answer, the standard analysis of repeated games, which imposes worst possible punishment path following *any* deviation (Abreu, 1988), answers "yes". However, in the context such as credit relationships where certain actions are arguably better, when deviations to better actions/histories occur, reducing the continuation value of the deviating or non-deviating player is hardly justified, morally or economically, as argued in the main text.

[5] Note that these two histories only differ in the last period and the new history has larger loan at that period.

increase the loan size as long as the borrower's no-deviation constraint is not binding, which implies that those constraints must be binding at all dates on path.

Given the above implication, we can explain the intuition for our main result, which shows that the loan sequence in any orthodox social equilibrium is strictly increasing. On the one hand, when past histories are unobservable, the high re-matching probability undermines the punishment power of the threat of terminating a relationship; so in order to induce repayment, there has to be some additional cost of starting a new relationship. This additional cost is reflected in the fact that the value of a relationship is increasing in its length, so that restarting a new relationship is worse than staying in the current relationship. On the other hand, the lender knows that the longer the borrower has stayed in a given relationship, the higher the value of this relationship to the borrower will be. Since the value of becoming unmatched is constant, the cost inflicted upon the borrower by terminating the current relationship becomes larger as time goes on; that is, the cost of defaulting increases over time. Then, since the lender has an incentive to exploit the borrower's no-deviation constraints, the loan size she offers also increases over time.

The rest of the paper is organized as follows. Section 2 sets up the model and states the main results; Section 3 discusses the intuition for our results, multiplicity of equilibria, and extension to mixed strategies. Section 4 concludes. All proofs are in the Appendix.

Related Literature. This paper contributes to the literature on trust building by showing that, in a repeated lender-borrower model, high likelihood of anonymous re-matching and sufficiently patient players—without assuming efficiency, multiple types and/or contractual commitment—are enough to deliver a unique prediction of strictly increasing loans over time in non-trivial equilibria.

The paper closest to ours is Datta (1996). He considers a special case of our model with linear payoffs and immediate re-matching, and shows that the *value of the relationship*—defined as the discounted sum of current and future loans—is nondecreasing in *efficient* social equilibria. This paper offers a sharper and more testable prediction in a more general setup where we allow for probabilistic re-matching and richer payoff structures. Compared to the result in Datta (1996), our prediction of strictly increasing loans is sharper because nondecreasing values still permit non-monotone loans over time; it is also much easier to test because the value of a relationship is much harder to observe than the level of interactions each period.[6] Moreover, we propose an equilibrium refinement to address what should happen after an unexpected higher loan is repaid. It rules out unreasonable equilibria supported by "unjustifiable punishments". Unlike Datta (1996)), since this refinement does not impose efficiency, our result is robust to inefficiency, whereas Datta's weaker characterization hinges on the efficiency assumption. Kranton (1996) studies a repeated game with re-matching where the stage game is of *simultaneous* moves with incentives similar to a prisoners' dilemma. She characterizes the cooperation levels in efficient equilibria to be "starting small" in the first period and reaching the efficient level from the second period on. Social equilibria of this type are ruled out by our refinement, and are arguably unreasonable in credit relationships where the stage game is of *sequential* moves (see the last paragraph of Section 3.1 for a detailed discussion).

Two other explanations for gradualism can be found in the literature. One strand of explanations combines moral hazard with incomplete information by introducing multiple types (usually reflected in levels of patience) to one side or both sides of a relationship. The reason for starting small in such environments is that, when the history of cooperation is longer, Bayes' rule implies that the probability of the other party being the "good" type is higher, so the optimal level of interaction increases over time. Ghosh and Ray (1996) study a repeated game with both incomplete information (two types) and re-matching, where the stage game is of simultaneous moves with incentives similar to a prisoners' dilemma. Their characterization of the evolution of cooperation levels is similar to Kranton (1996): "starting small" only in the very first period. Watson (1999, 2002) studies a general model of long-term relationships with incomplete information, and characterizes the level of interactions to be increasing over time under certain refinements. Kartal et al. (2015) studies a repeated trust game with two types of "receivers" (without re-matching), and finds a similar increasing pattern of trust levels. Applications of this idea, among others, include Rauch and Watson, (2003) in the context of trading between a supplier and a less informed buyer, as well as Araujo et al. (2016) and Antràs and Foley (2015) in the context of trade credit. The driving force of gradualism in these settings is the gradual learning of the other party's type.

Another strand of explanations shows up in the literature on self-enforcing contracts. Ray (2002) studies a very general repeated moral hazard problem without re-matching or incomplete information, and proves a similar result of "starting small" for *efficient* self-enforcing contracts. That is, in all efficient self-enforcing contracts, the continuation payoffs move over time in the direction of the agent who has an incentive to renege in the stage game. The main idea there is to apply the backloading argument up to a point: postponing higher rewards to the agent can keep the agent's current value the same while relax the agent's current incentive constraint (because future becomes more valuable), so that the agent can be incentivized to work harder or repay more today, which increases the principal's value and thus improves efficiency. Applications of this general insight include, among others, Lazear (1981) and Thomas and Worrall (1988) in the context of wage contracts, Thomas and Worrall (1994) in the context of foreign direct investment with threat of expropriation, and Albuquerque and Hopenhayn (2004) in the context of credit relationships. The driving force of gradualism in these settings is the interaction between efficiency and self-enforcing constraint.

[6] For example, our result can be falsified if the *observed* history of loans is not monotone. However, this simple test does not work for Datta's result because non-monotone loans could still be consistent with monotone values.

2. Model

2.1. Model setup

Consider a lender and a borrower in a society with a group of lenders and a group of borrowers. Both are infinitely lived. Time is discrete and starts from 0. The lender's and the borrower's utility functions are the discounted sums of their expected period payoffs. Specifically, let δ be the common discount factor and let $\mathbf{y}^L = \{y_0^L, y_1^L, \ldots\}$ be the sequence of expected period payoffs to the lender. The utility function of the lender at time t is:

$$V_t^L(\mathbf{y}^L) = \Sigma_{i=0}^{\infty} \delta^i y_{t+i}^L. \tag{1}$$

Similarly, let $\mathbf{y}^B = \{y_0^B, y_1^B, \ldots\}$ be the sequence of expected period payoffs to the borrower and define:

$$V_t^B(\mathbf{y}^B) = \Sigma_{i=0}^{\infty} \delta^i y_{t+i}^B. \tag{2}$$

The stage game takes the following form. Each period t is divided into 3 subperiods.[7] At t^0, the lender chooses the size of the loan, $L_t \in [0, L^*]$, granted to the borrower. At t^1, the borrower chooses whether to repay or default. At t^2, the lender and the borrower simultaneously choose whether or not to continue their relationship. The relationship continues if and only if both of them choose to do so. We assume away exogenous separation only for expository purposes.[8]

If the relationship continues, in the next period they repeat the stage game as described; if it is terminated, each party enters the next period as an unmatched lender/borrower. In each period, an unmatched agent will be anonymously matched with a new partner with an exogenous probability $\lambda \in [0, 1]$. If matched, she starts the stage game with the new partner in this period[9]; otherwise, she earns a payoff of 0 in this period and enters the next period as an unmatched agent.

The history of agents' actions in the *current* relationship is common knowledge to the borrower and the lender forming this relationship; histories of all past relationships of any party are unobservable,[10] so newly matched partners effectively restart from the very beginning of the game.

The payoffs in each period are determined by the loan size offered by the lender and the repayment decision of the borrower. Specifically, for a relationship in its period t,

$$y_t^L = \begin{cases} R(L_t), & \text{if repayment happens;} \\ -L_t, & \text{if default happens;} \end{cases} \tag{3}$$

$$y_t^B = \begin{cases} C(L_t), & \text{if repayment happens;} \\ D(L_t), & \text{if default happens.} \end{cases} \tag{4}$$

To summarize, a lender-borrower game with anonymous re-matching depends on the following elements: the discount factor δ, the re-match probability λ, the loan upper bound L^*, function R for the lender, and functions C and D for the borrower. For the sake of exposition, when no confusion arises we denote the game by $G(\delta, \lambda)$, omitting the dependence with respect to the R, C, D functions and the loan upper bound L^*.[11]

Assumption 1.

- $R(\cdot)$, $C(\cdot)$, $D(\cdot)$ are continuous and strictly increasing, and $R(0) = C(0) = D(0) = 0$;
- $\Delta(L) \equiv D(L) - C(L) > 0$ for all $L \in (0, L^*]$, and is strictly increasing;
- There exist $\underline{\alpha}, \bar{\alpha}$, such that $0 < \underline{\alpha} \le \bar{\alpha} < 1$, and $\underline{\alpha} D(L) < C(L) < \bar{\alpha} D(L)$ for all $L \in (0, L^*]$.

Assumption 1 requires that the borrower's payoffs from both default and repayment increase with loan size; in addition, the gains from default, $\Delta(L)$, are positive and also increase with loan size, capturing the borrower's myopic incentive to default. For the lender, if repayment occurs, a larger loan generates a higher period payoff; meanwhile, if default occurs, the lender bears a cost that increases with loan size. The last part of Assumption 1 ensures that $\frac{C(L)}{D(L)}$ is always bounded away from 0 and 1.[12] Note that L_t in general can be viewed as the level of trust offered by the lender. The higher the level

[7] Note that t is understood as the date in the *current* relationship; as explained later, since we focus on equilibria where every new relationship is a restart of the same relationship, it is sufficient to study one specific relationship.

[8] All results still hold if Nature terminates the relationship w.p. β in each period, with the only change being that the discount factor used in the thresholds should be $\delta(1-\beta)$, instead of δ. See Online Appendix and the SSRN version of this paper for details.

[9] Once matched, a previously unmatched lender (borrower) still plays the role of a lender (borrower) in the new relationship.

[10] Note that this is equivalent to saying that the histories of all past relationships of the *other* party are unobservable, and the agents also do not base their decisions on their own history of actions in past relationships, which in principle are observable to themselves.

[11] The results of this paper, Propositions 1 and 2, still hold even when $L^* = \infty$, i.e. when there is no upper bound on loans. However, if the upper bound L^* is random due to some exogenous shocks to the lender or the borrower, then in some orthodox social equilibria defined in Section 2.2, loans may not be strictly increasing even if the conditions in Proposition 1 are satisfied. as the exogenous shocks can force the lender to have to lend less than the previous period.

[12] If C and D are differentiable at 0, the last part of Assumption 1 is equivalent to $0 < C'(0) < D'(0) < \infty$.

of trust, the higher the payoffs for both parties given cooperation (repayment); however, a higher L_t also leads to a higher temptation to defect (default), as captured by the assumption that $\Delta(L) \equiv D(L) - C(L)$ is strictly increasing.

Example. Borrowing capital for production

Suppose that the borrower needs capital for production. Their payoffs can be modeled as follows:

$$y_t^L = \begin{cases} rL_t, & \text{if repayment happens;} \\ -L_t, & \text{if default happens;} \end{cases}$$

$$y_t^B = \begin{cases} F(L_t) - (1+r)L_t, & \text{if repayment happens;} \\ F(L_t), & \text{if default happens.} \end{cases}$$

where F is a production function of capital, and r is the fixed interest rate. Assumption 1 is satisfied if $F(L) - (1+r)L$ is strictly increasing in L and $1 + r < F'(0) < \infty$.

Notice that in the unique subgame perfect Nash equilibrium of the stage game (without termination decision), the lender chooses a loan size of 0 and the borrower always defaults on any positive loan. In the repeated game setting, we would like to focus on the equilibria in which the strategies depend only on the history of the current relationship, and the same strategy profile is played by all the agents in all relationships. That is, what agents do in a new relationship is an exact repetition of any prior relationship. Given our focus, it is without loss to look at the game of a lender and a borrower in a single relationship while taking as given the continuation game (payoffs) on terminating the current relationship, which itself is determined in equilibrium.

2.2. Orthodox social equilibrium

We now define an equilibrium concept for our game. Let $L_t \in [0, L^*]$ be the loan size chosen at t^0; let $d_t \in \{0, 1\}$ denote the borrower's defaulting decision at t^1, s.t. $d_t = 1$ iff default happens at t; let $f_t \in \{0, 1\}$ and $g_t \in \{0, 1\}$ denote the lender's and the borrower's decisions on continuing the relationship at t^2, s.t. their relationship is terminated at t iff $f_t g_t = 0$. Denote by a_t the outcome of these decisions at period t; that is, $a_t = (L_t, d_t, f_t, g_t)$. The history at each node is denoted by:

$$h(t^0) = \{a_0, a_1, \ldots a_{t-1}\}, \text{ where } h(0^0) = \emptyset;$$
$$h(t^1) = h(t^0) \cup \{L_t\};$$
$$h(t^2) = h(t^1) \cup \{d_t\}.$$

Let $H(t^i)$ be the collection of all possible histories at t^i, for $i = 0, 1, 2$.

A (pure) strategy of the lender $l = \{l_0, l_1, \ldots\}$ consists of a sequence of decision rules that maps each information set to her decision at that node. Specifically, $l_t = (\tilde{L}_{t0}, \tilde{f}_{t2})$, where $\tilde{L}_{t0} : H(t^0) \to [0, L^*]$ and $\tilde{f}_{t2} : H(t^2) \to \{0, 1\}$, s.t. $L_t = \tilde{L}_{t0}[h(t^0)]$ and $f_t = \tilde{f}_{t2}[h(t^2)]$. Similarly, a (pure) strategy of the borrower $b = \{b_0, b_1, \ldots\}$ is defined as $b_t = (\tilde{d}_{t1}, \tilde{g}_{t2})$, where $\tilde{d}_{t1} : H(t^1) \to \{0, 1\}$ and $\tilde{g}_{t2} : H(t^2) \to \{0, 1\}$, s.t. $d_t = \tilde{d}_{t1}[h(t^1)]$ and $g_t = \tilde{g}_{t2}[h(t^2)]$. Notice that given a strategy profile $\{l, b\}$, we are not yet able to compute the payoff of each agent, if according to $\{l, b\}$ a relationship is terminated at some date. This is because the continuation values after termination of a relationship depend on equilibrium payoffs, but we have not solved for them; therefore, the equilibrium concept involves a fixed point between re-matching values and equilibrium payoffs.

Let $\bar{V}^L$ and $\bar{V}^B$ be the re-matching values (i.e. values of a newly-matched relation) for the lender and the borrower, respectively. Note that the continuation values of an unmatched lender and borrower are given by $\lambda'\bar{V}^L$ and $\lambda'\bar{V}^B$, where $\lambda' = \frac{\lambda}{1-(1-\lambda)\delta}$.[13] Given a strategy profile (l, b), we are able to trace out a sequence of decisions on the equilibrium path, $\{a_t\}_t$, where $a_t = (L_t, d_t, f_t, g_t)$. Let $T(l, b)$ be the date at which the relationship is terminated according to (l, b), i.e. $f_{T(l,b)} g_{T(l,b)} = 0$, and for all $t < T(l, b)$, $f_t g_t = 1$. From (1) and (2), we can write:

$$V_t^L(l, b, \bar{V}^L, \bar{V}^B) = \sum_{i=0}^{T(l,b)-t} \delta^i [(1 - d_{t+i}) R(L_{t+i}) - d_{t+i} L_{t+i}] + \delta^{T(l,b)-t+1} \lambda' \bar{V}^L, \tag{5}$$

$$V_t^B(l, b, \bar{V}^L, \bar{V}^B) = \sum_{i=0}^{T(l,b)-t} \delta^i [(1 - d_{t+i}) C(L_{t+i}) + d_{t+i} D(L_{t+i})] + \delta^{T(l,b)-t+1} \lambda' \bar{V}^B. \tag{6}$$

In (5), a lender's payoff is the discounted sum of her current and future period payoffs. At $t+i$, her period payoff is $R(L_{t+i})$ if repayment happens and $-L_{t+i}$ otherwise; in addition, at $T(l, b)$ when the relationship is terminated, she will also get the discounted continuation value of an unmatched agent. In (6), a borrower's payoff has the same structure as that of a lender, where at $t+i$ the borrower gets $C(L_{t+i})$ if repayment happens and $D(L_{t+i})$ otherwise, plus the present value of her continuation payoff when the relationship is terminated.

Definition 1. A **social equilibrium** consists of a strategy profile (l, b) and re-matching values $(\bar{V}^L, \bar{V}^B)$, such that

[13] To see this, let $\hat{V}^B$ be the continuation payoff to the unmatched borrower. We have $\hat{V}^B = \lambda\bar{V}^B + (1-\lambda)\delta\hat{V}^B$, so that $\hat{V}^B = \frac{\lambda}{1-(1-\lambda)\delta}\bar{V}^B$.

(i) Gvien equilibrium payoffs $\bar{V}^L$ and $\bar{V}^B$, l and b are sequentially rational w.r.t. each other;
(ii) $\bar{V}^L = V_0^L(l, b, \bar{V}^L, \bar{V}^B)$, $\bar{V}^B = V_0^B(l, b, \bar{V}^L, \bar{V}^B)$.

Part (i) of Definition 1 is just the standard requirement of subgame perfection, while part (ii) captures our fixed point requirement for re-matching values. We call it social equilibrium because part (ii) implicitly assumes that every pair in the society plays such a strategy profile in every relationship. We focus on pure-strategy equilibria in most of this paper, and discuss the extension to mixed strategies in Section 3.3.

We further restrict our attention to orthodox social equilibria, which are social equilibria such that: (i) an ongoing relationship is terminated on default; (ii) a better loan-repayment history is followed by weakly higher continuation values for the lender and the borrower. Formally, given a strategy profile (l, b), let $V_t^L : H(t^0) \to \mathbb{R}$ and $V_t^B : H(t^0) \to \mathbb{R}$ be the induced continuation value functions for the lender and the borrower at the beginning of each period, which are maps from the set of histories at t^0 to real numbers. These specify the remaining values of the *current* relationship to the lender and the borrower following any history.

Definition 2. A social equilibrium strategy profile (l, b) is **orthodox**, if

(i) For any given t and any $h(t^2) \in H(t^2)$ s.t. $d_\tau = 1$ for some $\tau \le t$, we have $\tilde{f}_{t^2}[h(t^2)] = 0$;
(ii) For any given t and any $h(t^0)$, $h'(t^0) \in H(t_0)$ s.t. for all $\tau < t$, $L'_\tau \ge L_\tau$ (with equality for all $\tau < t-1$), $d'_\tau = d_\tau = 0$ and $f'_\tau g'_\tau = f_\tau g_\tau = 1$, we have $V_t^L[h'(t^0)] \ge V_t^L[h(t^0)]$ and $V_t^B[h'(t^0)] \ge V_t^B[h(t^0)]$.[14]

Part (i) of Definition 2 requires that the lender terminates the relationship if the borrower has defaulted before. This requirement is standard in the literature (see also Datta, 1996; Ghosh and Ray, 1996; 2016; Kranton, 1996, etc.); it simplifies the analysis by imposing a fixed, in fact worst,[15] punishment on the borrower's default.

Part (ii) of Definition 2 needs more justifications. It says that at the beginning of any period t, if we consider two histories such that the borrower has made repayments at all dates in both of them, and the loan sizes only differ in the last period (i.e. $t-1$), then the continuation values for both players following the history with a higher last-period loan should be weakly higher.[16] This restriction is motivated by the following idea. One can view an equilibrium in a repeated game as an implicit contract between the players. Such a contract specifies each player's continuation value at every history. Let us fix a period t and look at two different histories at the beginning of period t—$h(t^0)$ and $h'(t^0)$—among which $h(t^0)$ is on the equilibrium path. To sustain $h(t^0)$ as part of the equilibrium path, any deviation that benefits the deviating player in the current period has to be prevented via sufficiently lowering her continuation value, which is achieved by changing future course of play. But such a punishment imposed on the deviating player is not "morally justified" if the deviation unambiguously improved the other player's payoff. In the end, on what ground should a person be "punished" if her actual behavior—compared to what she was expected to do—strictly benefits another party? Moreover, it is not even "economically justified" if the punishment is costly to the non-deviating player (who is the "carrier" of the punishment). In the context of credit relationships, if the only difference between $h(t^0)$ and $h'(t^0)$ is that $h'(t^0)$ has a higher last-period loan (with all repayments made), it is fair to claim that both the lender and the borrower have unambiguously benefited from the deviation to $h'(t^0)$.[17] The previous argument then implies that neither player should receive a strictly lower continuation value after history $h'(t^0)$ than after $h(t^0)$, for otherwise it would be either morally unjustified or economically unjustified (or both).

The idea behind Part (ii) is related to the notion of "justifiable punishments" (Aramendia and Wen, 2014), which is a refinement defined for normal-form repeated games with perfect monitoring. Using a reasoning similar to above, they also put a monotone restriction on the continuation values following deviations that benefit the other party.[18]

Finally, note that the definition of orthodox social equilibrium does not require efficiency. So the main result of this paper, which establishes strictly increasing loans in all non-trivial orthodox social equilibria, holds for inefficient equilibria as well.

[14] A weaker requirement that a better loan-repayment history leads to weakly higher continuation values to both players from the *rest of the game* (including both the remaining values from the current relationship, and the values from the possibility of re-matching if the current relationship is terminated later) would deliver exactly the same results.

[15] The borrower cannot be made worse than being unmatched, as she can unilaterally terminate the relationship.

[16] To be sure, we only make comparisons at subperiod t^0, where a history $h(t^0)$ consists of a full set of actions (loan size, defaulting decision, continuation decisions) for each period from 0 to $t-1$. We do not make comparisons or impose such restriction at other subperiods. Moreover, it is perfectly allowed if one of the two histories has a *strictly* higher last-period loan but the continuation values following the two histories are the same. For example, this will be the case if the players simply ignore deviations to higher but repaid loans and keep the future course of player unchanged.

[17] The history $h'(t^0)$ may require deviations from both players, in which case both are deviating players.

[18] Our refinement is different from Aramendia and Wen (2014) in several aspects. First, the monotonicity restriction in Aramendia and Wen (2014) is only imposed on the non-deviating player's continuation value, while we require that the deviating player is also not punished if the deviation benefits the other player. Second, their refinement is only defined for normal-form repeated games with perfect monitoring, whereas in our context the stage game is of sequential moves. (Even though we can transform an extensive-form stage game into a normal-form one, the assumption of perfect monitoring would then become problematic because the strategy of the borrower in the stage game—a function mapping each loan size to a repayment decision—is not fully observed.) Moreover, we apply the logic of "justifiable punishments" only at the beginning of each period (i.e. only at t^0, not t^1 or t^2).

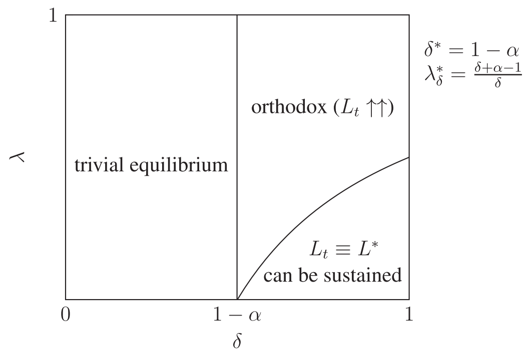

Fig. 1. Equilibrium characterization for linear payoff functions.

2.3. The structure of orthodox social equilibrium

Note first that a trivial (orthodox) social equilibrium always exists, in which the lender always offers a loan size of 0 and the borrower defaults on any positive loan. Also observe that in any non-trivial social equilibrium, the relationship is never terminated on path.[19] This is because if it is terminated at date t of a relationship, then at that last period (of the relationship) the borrower will default on any positive loan; as a result, the loan size at date t must be 0. But then, at the end of the second-to-last period $t-1$, each agent has an incentive to terminate the relationship because both of them would prefer getting the values of an unmatched agent right away (which is positive because the equilibrium is non-trivial), rather than waiting for another period with a payoff of 0 and then getting such values. But this is a contradiction to the optimality of continuing the relationship at the end of $t-1$. One immediate implication is that in any non-trivial orthodox social equilibrium, the borrower never defaults on path, for otherwise a relationship would be terminated on default by definition of an orthodox social equilibrium.[20]

Now we state the main results of this paper.

Proposition 1. *Suppose that Assumption 1 holds. There exists a $\delta^* < 1$ s.t. the following holds: for all $\delta \in (\delta^*, 1)$, there exists a $\lambda^*_\delta < 1$ s.t. whenever $\lambda \in (\lambda^*_\delta, 1]$, the loan sequence $\{L_t\}_t$ is strictly increasing on the equilibrium path of any non-trivial orthodox social equilibrium of the game $G(\delta, \lambda)$.*

Proposition 2. *Suppose that Assumption 1 holds. Whenever $\delta \in (\delta^*, 1)$ and $\lambda \in (\lambda^*_\delta, 1]$, a non-trivial orthodox social equilibrium exists in the game $G(\delta, \lambda)$.*

Remark 1: Small re-matching probability/discount factor. One may wonder about the structure of orthodox social equilibria when the re-matching probability or the discount factor is smaller than their thresholds. It turns out that if we assume linear payoff functions (see Section 3.1, $\frac{C(L)}{D(L)} \equiv \alpha \in (0, 1)$ for all L), we can characterize equilibria for all parameter values.

As illustrated in Fig. 1, when $\delta < 1-\alpha$, only the trivial equilibrium ($L_t = 0$ for all t) exists. This often shows up in repeated games even without re-matching: when agents do not care enough about the future, they can only play the unique SPE of the stage game. On the other hand, when the agents are patient and the re-matching probability is low (i.e. the bottom right region of Fig. 1), the threat of terminating a relationship is strong enough to sustain a maximum loan level from the beginning. This is because re-matching in this case is so unlikely that losing the current relationship is nearly as bad as being thrown out of the market completely. Finally, notice that the threshold λ^*_δ increases with δ. This is intuitive: when the agents become more patient, it is easier to sustain $L_t \equiv L^*$ because the future cost of terminating a relationship is higher for a more patient borrower.

Therefore, an increasing sequence of loans will arise when agents are sufficiently patient while the re-matching probability is also high; the former ensures that the agents are able to sustain some positive level interactions instead of only playing the trivial equilibrium, while the latter creates the need for "starting small" because a constant loan size over time will directly result in "hit and run".

Remark 2: "Motonone values" restriction vs. "Monotone loans" result. One important requirement in the definition of an orthodox social equilibrium is that a better loan-repayment history does not strictly decrease the continuation value of the relationship to either party. We motivated this requirement using an argument of "justifiable punishments". But it is still fair to ask: does this monotone restriction on continuation values select equilibria with strictly increasing loan sizes by

[19] As pointed out in footnote 2.1, we can allow exogenous separation by Nature with certain probability at the end of each period, and our results still hold. In that case, a relationship is never terminated *by agents* on the equilibrium path.

[20] Note that we imposed the condition that default implies termination only when defining "orthodox social equilibrium". For a social equilibrium in general, although the relationship is never terminated by agents (as argued in the text), its definition allows default to occur on the equilibrium path.

assuming it? The answer is "no" because the monotonicity in the restriction and in the result are across different dimensions. The "monotone values" restriction is a condition that compares continuation values across different histories at *fixed t*; in contrast, the "monotone loans" result is about the evolution of loan sizes *over time*. In addition, trivial equilibria in which loan size is always 0 are not ruled out by such a refinement. In fact, if the re-matching probability is too low (i.e. if the condition in Proposition 1 is violated), there are orthodox social equilibria in which the loan sizes are strictly *decreasing* over time.[21] These equibria are not ruled out by our refinement per se; however, they are impossible when the re-matching probability is high, because the borrower would want to "hit and run" if loans were decreasing and re-matching is easy.

As we will illustrate in Section 3.1: technically, what the "monotone values" restriction buys us is a sequence of *binding* no-default constraints; economically, the real driving forces of the gradualism result in this paper is the high re-matching probability and justifiable punishments.

3. Discussion

3.1. An intuition for Proposition 1 with linear payoff functions

To understand the intuition behind Proposition 1, consider the case where payoff functions are linear in loan size as follows[22]:

$$y_t^L = \begin{cases} (1-\alpha)L_t, & \text{if repayment happens;} \\ -L_t, & \text{if default happens;} \end{cases}$$

$$y_t^B = \begin{cases} \alpha L_t, & \text{if repayment happens;} \\ L_t, & \text{if default happens.} \end{cases}$$

Let $\mathbf{L} = \{L_t\}_t$ be the sequence of loans on the equilibrium path of some non-trivial orthodox social equilibrium. As we observed earlier, a relationship is never terminated in any non-trivial social equilibrium. Therefore, we can write the lender's and the borrower's values at each date as:

$$V_t^L[(1-\alpha)\mathbf{L}] = \Sigma_{i=0}^{\infty}\delta^i(1-\alpha)L_{t+i},$$
$$V_t^B(\alpha\mathbf{L}) = \Sigma_{i=0}^{\infty}\delta^i\alpha L_{t+i}.$$

Note that the borrower's no-deviation (no-default) constraints are: for all t,

$$(1-\alpha)L_t \le \delta[V_{t+1}^B(\alpha\mathbf{L}) - \lambda' V_0^B(\alpha\mathbf{L})], \tag{7}$$

where the LHS is the current period gains from default, and the RHS is the future cost of default which is the difference between the value of continuing the relationship and the value of terminating the relationship. (Recall that λ' is derived in footnote 12.)

One implication of part (ii) of Definition 2 is that in any orthodox social equilibrium, (7) must hold at equality for all t such that $L_t < L^*$. That is, the lender has an incentive to increase the loan size as much as possible so that in each period, either the borrower's no-deviation constraint is binding, or the loan size reaches its maximum L^*. To see this, suppose that in some period t, (7) holds at strict inequality and $L_t < L^*$. Then in subperiod t^0 on the equilibrium path, the lender can consider deviating to $L_t' = L_t + \varepsilon < L^*$ s.t.

$$(1-\alpha)L_t' < \delta[V_{t+1}^B(\alpha\mathbf{L}) - \lambda' V_0^B(\alpha\mathbf{L})]. \tag{8}$$

Is this deviation profitable? Note first that such a deviation will not induce default in period t. This is because in subperiod t^1 when the borrower decides whether or not to repay, she will find:

$$(1-\alpha)L_t' < \delta[V_{t+1}'^B - \lambda' V_0^B(\alpha\mathbf{L})], \tag{9}$$

where $V_{t+1}'^B$ is the borrower's continuation value at $(t+1)^0$ (i.e. the beginning of date $t+1$) following the new history with her repayment for a larger loan L_t'. (9) holds because of (8) and $V_{t+1}'^B \ge V_{t+1}^B(\alpha\mathbf{L})$, where the latter condition follows from the fact that the loan-repayment history with loans $\{L_0, L_1, \dots L_{t-1}, L_t'\}$ and no default is better than that with $\{L_0, L_1, \dots L_{t-1}, L_t\}$ and no default. As a result of (9), deviating to L_t' would not induce default.

[21] For example, when "$\delta = 0.8, \alpha = 0.5, \boldsymbol{\lambda} = \mathbf{0.2}, L^* = 100$", $\{L_t\}_t = \{93.8, 77.4, 67.1, 60.7, \dots\}$ is sustained by an orthodox social equilibrium. In contrast to Proposition 1, our refinement cannot give a unique prediction on loan monotonicity when agents are patient (high δ) and re-matching is unlikely (low λ). Such an equilibrium may not be very reasonable though, because under these parameters $L_t \equiv L^*$ can also be sustained as an orthodox equilibrium, and offering L^* in each period is the *first best* scenario that generates the highest possible payoffs to both parties in this model. This example is just to show that the "monotone values" restriction itself does not select equilibria with increasing loans.

[22] This can be interpreted as a model of trade credit. The lender (as an exporter) cannot directly face the consumers in another country and has to sell the product via a borrower (as an importer). The borrower does not consume the good but resells them to the consumers and collects the revenue. Assume that the financing cost to the borrower is prohibitive so that it is impossible to require the borrower to pay any amount of money before it receives and resells the good. Thus in each period, the lender first ships the good of value L_t to the borrower; the borrower resells the good, collects the revenue and decides whether or not to pay back $(1-\alpha)L_t$. If repayment happens, the borrower gets αL_t and the lender gets $(1-\alpha)L_t$; if default happens, the borrower keeps all the revenue L_t, while certain additional cost is incurred to the lender so that she gets $-L_t$.

Knowing this, the lender's total payoff from time t by offering L'_t will be $(1-\alpha)L'_t + V'^{L}_{t+1}$, which is strictly larger than $(1-\alpha)L_t + V^L_{t+1}[(1-\alpha)\mathbf{L}]$ because $L'_t > L_t$ and $V'^{L}_{t+1} \geq V^L_{t+1}[(1-\alpha)\mathbf{L}]$, where the second inequality is due to the fact that the lender's value following a better loan-repayment history is also weakly higher. But this just implies that deviating to L'_t is profitable for the lender, a contradiction to $\{L_t\}_t$ being on the equilibrium path of some (non-trivial) orthodox social equilibrium.

Therefore, when the deviation to a better loan-repayment history is not punished, the lender will have an incentive to increase the loan size as much as possible, so that in each period either the borrower's no-deviation constraint is binding, or L_t reaches its maximum L^*. It turns out that the latter case can be ruled out when λ (the re-matching probability) is high, so the borrower's no-deviation constraint is binding at each date.

Now we can explain the intuition for our main result, which says that when the discount factor and the re-matching probability are high, the equilibrium loan sequence in any orthodox social equilibrium is strictly increasing. On one hand, when past histories are unobservable, the high re-matching probability undermines the punishment power of the threat of terminating a relationship; so in order to induce repayment, there has to be some additional cost of starting a new relationship, which here is reflected in the fact that V^B_t is strictly increasing over time. On the other hand, because the value of becoming unmatched is constant while V^B_t grows, the cost inflicted upon the borrower by terminating the current relationship is increasing over time. This means that the cost of defaulting increases over time. Since the lender has an incentive to make the borrower's no-deviation constraint bind in each period, she will finally offer a strictly increasing sequence of loans.

Another way to understand the result, though more technical, is to directly inspect the loan sequence. First, high re-matching probability implies that a non-trivial equilibrium loan sequence cannot be constant (or decreasing) over time, for otherwise the borrower would just default in the first period, run away, get rematched, and default again. As before, this calls for the value, as well as the loan size, to start small. But can it be that the loan size increases for a while and then stays constant starting from certain period $\tilde{t}$? This is the type of equilibrium showing up in Kranton (1996) and Ghosh and Ray (1996). It is ruled out by our refinement because for any loan sequence of this type, the *future* interactions after period $\tilde{t}-1$ and after period $\tilde{t}$ are exactly the same (i.e. both consist of a constant loan size forever). That is to say, the borrower's costs of defaulting at $\tilde{t}-1$ and $\tilde{t}$, RHS of (7), are the same. But then, since by construction the loan size at $\tilde{t}-1$ is smaller than that at $\tilde{t}$, this implies that $L_{\tilde{t}-1} < L^*$ and the borrower's no-deviation constraint is not binding at $\tilde{t}-1$, a contradiction to what we have explained before (i.e. no-deviation constraint should be binding as long as $L_t < L^*$).[23] These arguments, together with the fact that we can rule out other possibilities (e.g. the sequence decreases once in a while, or it stays constant for a while and then increases, etc. See Lemma 3 in the Appendix), establishes that any non-trivial equilibrium loan sequence must be strictly increasing over time when the re-matching probability is high.

3.2. *Equilibrium non-uniqueness*

Though our refinement yields a unique prediction regarding loan monotonicity for interesting parameter values, it is unsurprising that equilibrium is not unique in this repeated game. We use the special case of linear payoff functions (as in Section 3.1) to illustrate one force that leads to the multiplicity of orthodox social equilibrium. We then discuss the same multiplicity issue in Datta (1996).

Consider the following parametrization:

$$\delta = 0.8, \alpha = 0.5, \lambda = 1, L^* = 100.$$

Under these parameters, the following two loan sequences can both be sustained by some orthodox social equilibria:

$$\{L_t\}_t = \{37.5, 60.9, 75.6, 84.7, 90.5, \ldots\};$$
$$\{L'_t\}_t = \{18.8, 30.5, 37.8, 42.4, 45.2, \ldots\}.$$

In our construction, $L'_t = \frac{1}{2}L_t$ for all t (before rounding up). In fact, with linear payoff functions, if $\{L_t\}_t$ is an equilibrium loan sequence, so is $\{\beta L_t\}_t$ for any $0 < \beta < 1$. This is because, moving from $\{L_t\}_t$ to $\{\beta L_t\}_t$, both sides of the borrower's no-default constraint (7) will be scaled by β; so it holds for the new sequence if and only if it holds for the original one.[24] Nevertheless, as Proposition 1 states, all these sequences are strictly increasing.

Datta (1996) studies the linear payoff environment where the re-matching probability is 1. He focuses on the efficient equilibria within the class of social equilibria that entail no default on path, which he called *maximal social equilibria*. His

[23] We reiterate the exact reason why this type of social equilibrium is unreasonable in the context of credit relationships. The reason lies in the strategy profile that supports such a loan sequence. We have seen that the borrower's no-deviation constraint is slack at $\tilde{t}-1$ on path. But the fact that it is equilibrium implies that the lender does not want to increase the loan size at $\tilde{t}-1$ even by just a little. This means that either an increase in loan size causes default, or even after the repayment the lender is still worse off. In the former case, the future value of the relationship to the borrower must be lowered after she repays a higher loan; in the latter case, the future value of the relationship to the lender must be lowered after she offers a higher loan followed by repayment. In either case, had a better (off-equilibrium) loan-repayment history been reached, at least one agent is "punished". This is the unreasonable feature suffered by *any* strategy profile that supports such a loan sequence that increases first and then stays constant.

[24] With nonlinear payoff functions, this construction by scaling each loan size by β is no longer valid for creating another equilibrium loan sequence. But depending on the exact function forms of C and D, reducing *all* loans by some (carefully chosen) amounts can generate a new equilibrium loan sequence.

main result is that the *value* sequences (of both parties) in any maximal social equilibrium are nondecreasing. While his prediction is weaker than ours, his refinement does not guarantee equilibrium uniqueness either. Under the same parameter values, both $\{L_t\}_t$ defined above and $\{L''_t\}_t$ defined below can be sustained by some maximal social equilibria[25]:

$$\{L''_t\}_t = \{35, 25, 100, 100, 100, \ldots\}.$$

Note that this loan sequence is non-monotone; in fact, Proposition 1 implies that most of the maximal social equilibria (except the orthodox ones) entail "unjustifiable punishments".[26]

In the linear environment considered here, the sequence $\{L_t\}_t = \{37.5, 60.9, 75.6, 84.7, 90.5, \ldots\}$ is the unique equilibrium loan sequence that is both orthodox and maximal.[27] In nonlinear environments, even if we focus on the *maximal orthodox* social equilibrium, uniqueness is still not guaranteed. The main contribution of this paper is to find reasonable conditions that deliver a unique prediction of strictly increasing loans, despite multiplicity of equilibria.

Finally, it is worth pointing out that the key technical condition which drives our result is the binding no-default constraints. As discussed in Section 3.1, this condition is implied by the requirement of "justifiable punishments". This paper shows that such a requirement is enough to generate a unique prediction of strictly increasing loans, even without imposing efficiency. Datta (1996) is unable to make a prediction regarding loan monotonicity because his analysis allows for "unjustifiable punishments" (and thus non-binding no-default constraints); and even after efficiency is imposed, only a weaker prediction regarding values can be made.

3.3. Extension to mixed strategies

The preceding analysis has focused on pure-strategy equilibria, in which the loan amount, defaulting and termination decisions are deterministic in each period. In this subsection, we will first argue that in equilibria where the relationship is never terminated and default never occurs, randomizing over loan amounts is not compatible with the notion of orthodox social equilibrium. Next, we will show that if payoffs are linear in loan and if the borrower's defaulting decision breaks even in favor of the lender, then in any orthodox social equilibrium, indeed the relationship is never terminated and default never occurs, even if mixed strategies (in termination decisions) are allowed.

Note first that the definition of orthodox social equilibrium can be readily extended to accommodate mixed strategy by interpreting the future values after a particular history as the *expected* future values.[28] Then, let us take any orthodox social equilibrium that has no default and no termination on the equilibrium path. Suppose that at the beginning of period t with on-path history $h(t^0)$, equilibrium behavior involves mixing between two loan amounts, L_t and L'_t, s.t. $L_t < L'_t$. Since the lender is indifferent between L_t and L'_t, it means that

$$R(L_t) + V^L_{t+1}[h((t+1)^0)] = R(L'_t) + V^L_{t+1}[h'((t+1)^0)],$$

where the two histories at the beginning of $t+1$, $h((t+1)^0)$ and $h'((t+1)^0)$, only differ in their last-period (period t) loans. Since $L_t < L'_t$, our refinement requires that $V^L_{t+1}[h((t+1)^0)] \le V^L_{t+1}[h'((t+1)^0)]$, but this is a contradiction to the indifference condition, as R is strictly increasing. This means that if default and termination never occur, randomizing over loan amounts is not compatible with the notion of orthodox social equilibrium.

Now suppose that:

- payoffs are linear in loan (as in Section 3.1);
- the borrower's defaulting decision breaks even in favor of the lender; i.e. she does not default when indifferent.[29]

In any mixed-strategy non-trivial orthodox social equilibrium, let us first show that default never happens on the equilibrium path. To see this, suppose (by contradiction) that default happens on path. Let t be the first period in which the borrower defaults on a positive loan. Since the relationship is terminated at the end of that period after default (as required by Definition 2), the lender can do strictly better by offering 0 loan in that period and then terminate the relationship. So a profitable deviation exists for the lender.

[25] To see that $\{L''_t\}_t = \{35, 25, 100, 100, 100, \ldots\}$ is maximal (i.e. efficient among the class of social equilibria that entail no default on path), note first that for any social equilibrium without default on path, $V^B_0 \le V^{*B}_0$, where V^{*B}_0 solves $(1-\alpha)L^* = \delta(\frac{\alpha L^*}{1-\delta} - \lambda' V^{*B}_0)$. This is because if there is a social equilibrium without default s.t. its loan sequence on path is $\{\tilde{L}_t\}_t$ and $\tilde{V}^B_0 > V^{*B}_0$, without loss we have $\limsup \tilde{L}_t = L^*$ due to linearity of payoff functions. But then, for $\tilde{L}_t$ close enough to L^*, we have $(1-\alpha)\tilde{L}_t > \delta(\frac{\alpha \tilde{L}_t}{1-\delta} - \lambda' \tilde{V}^B_0)$, a contradiction to the requirement that the borrower does not default on path. It can be checked that $\{L''_t\}_t = \{35, 25, 100, 100, 100, \ldots\}$ achieves V^{*B}_0 and satisfies the borrower's incentive constraints under our parametrization, so it is maximal.

[26] That is, at some date, if the lender unexpectedly increases the loan size and the borrower repays, the value of the relationship to at least one party will be strictly reduced.

[27] $\{L_t\}_t = \{37.5, 60.9, 75.6, 84.7, 90.5, \ldots\}$ is constructed using the algorithm proposed in the proof of Proposition 2 by setting the limit $L = L^*$. To see that $\{L_t\}_t$ is the unique loan sequence that is both orthodox and maximal, note first that by construction it can be supported by an orthodox social equilibrium using the strategy profile proposed in the proof of Proposition 2. Note also that by construction, $(1-\alpha)L_t = \delta(\frac{\alpha L_t}{1-\delta} - \lambda V^B_0)$, for all t. Since $L_t \to L^*$, we have $(1-\alpha)L^* = \delta(\frac{\alpha L^*}{1-\delta} - \lambda V^B_0)$; therefore $V^B_0 = V^{*B}_0$, where V^{*B}_0 is defined in footnote 3.2. This implies that such an orthodox social equilibrium is also maximal. Finally, the uniqueness of $\{L_t\}_t$ follows from the fact that it is the unique loan sequence that satisfies (10) and converges to L^* (see the proof of Proposition 2), and any other such sequence converging to $L < L^*$ generates a lower V^B_0.

[28] The expectation is taken over possible future consumption streams.

[29] This essentially assumes that the borrower's defaulting decision is deterministic, while allowing for mixing in loan sizes and termination decisions.

Next, we argue that the relationship is never terminated on path. Note that linearity and no default on path imply that the borrower's value is a fixed proportion $\frac{\alpha}{1-\alpha}$ of the lender's value. On the equilibrium path, let t be the first period in which the relationship is terminated with positive probability. This implies that the borrower's value (on path) at the beginning of next period $t+1$ must be weakly lower than $\lambda' V_0^B$.[30] As a result, the borrower will default on any positive loan that leads to termination with positive probability. So in this equilibrium, the history $h(t^2)$ on path after which termination is possible must have 0 loan in period t. But then, the lender should terminate the relationship one period earlier at the end of $t-1$, as she prefers getting the value of an unmatched lender right away (which is positive because the equilibrium is non-trivial) to waiting for another period with a payoff of 0 and then getting such a value. This is a contradiction to t being the first period in which the relationship is terminated with positive probability.[31]

To summarize, we have argued that: (i) for any orthodox social equilibrium in mixed strategies where termination and default never occur on path, loan amounts must be deterministic; (ii) if payoffs are linear in loan and the borrower repays when indifferent, then all orthodox social equilibria in mixed strategies involve no termination and no default on path, and thus deterministic loan amounts. So Propositions 1 and 2 apply to these cases.

4. Conclusion

This paper studies a lender-borrower game in a pure moral hazard environment with anonymous re-matching. The main result states that as long as the discount factor and the re-matching probability are above certain thresholds, the size of loans along the equilibrium path of any orthodox social equilibrium is strictly increasing over time. This characterization gives a formal argument that qualifies the possibility of anonymous re-matching as a driving force of gradualism in long-term relationships. Certainly, re-matching is not the only reason for gradualism, and type uncertainty, among other things, is also a reasonable and important driving force for such phenomena. However, given that in reality it is indeed costly to acquire the past history information of the other party in a new relationship, this paper provides insights from one specific aspect into understanding gradualism in long-term relationships, especially credit relationships.

Appendix A

We present the proofs of Propositions 1 and 2 in this Appendix. Throughout Appendix, we assume that Assumption 1 holds.

Lemma 1. *In any non-trivial social equilibrium, a relationship is never terminated by the agents; in addition, in any non-trivial orthodox social equilibrium, there is no default on the equilibrium path.*

Proof. The proof follows the lines of the first paragraph of Section 2.3. □

Lemma 2. *Let (l, b) be an orthodox social equilibrium, and $\mathbf{L} = \{L_t\}_t$ be the sequence of loan sizes on its equilibrium path. We have that for all t:*

$$\begin{aligned} & L_t = \max L \\ \text{s.t. } & L \le L^*, \\ & \Delta(L) \le \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]. \end{aligned} \tag{10}$$

where $V_t^B(\mathbf{L}) = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i})$.

Proof. By Lemma 1, in any orthodox social equilibrium, a relationship is never terminated by the agents and there is no default on path; therefore, we can write the borrower's continuation value on path at the beginning of each period as $V_t^B(\mathbf{L}) = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i})$.

Note also that the borrower's no-default constraint at t can be written as

$$\Delta(L_t) \le \delta\left[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})\right].$$

where the LHS is the borrower's current gains from defaulting on L_t at t, and the RHS is the present value of her future cost from defaulting, i.e., the difference between the value of continuing the relationship by repaying and the value of terminating the relationship by defaulting.

Now we prove (10) by contradiction. Let (l, b) be an orthodox social equilibrium s.t. (10) does not hold. Let t be the first period that (10) fails. Since $\Delta(\cdot)$ is strictly increasing, either $\Delta(L_t) > \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$, or $L_t < L^*$ and $\Delta(L_t) < \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$.

[30] To see this, note that if the relationship is terminated (with positive probability) because of the borrower's termination, we must have $V_{t+1}^B \le \lambda' V_0^B$; if it is because of the lender's termination, we must have $V_{t+1}^L \le \lambda' V_0^L$. Since linearity implies that $V_{t+1}^B = \frac{\alpha}{1-\alpha} V_{t+1}^L$ and $V_0^B = \frac{\alpha}{1-\alpha} V_0^L$, we again have $V_{t+1}^B \le \lambda' V_0^B$. Note that this argument is no longer valid if payoffs are not linear in loan.

[31] More precisely, this is a contradiction unless $t = 0$; but if the relationship is terminated with positive probability at $t = 0$, then the lender's initial value satisfies $V_0^L = 0 + \delta\lambda' V_0^L$, so that $V_0^L = 0$. But this can happen only in a trivial equilibrium.

If $\Delta(L_t) > \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})]$, since there is no default on the equilibrium path and the no-default constraint is violated, the borrower will be better off by a one-shot deviation to default at time t. This is a contradiction to (l, b) being mutual perfect best responses.

If $L_t < L^*$ and $\Delta(L_t) < \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})]$, choose L'_t s.t. $L_t < L'_t \le L^*$ and $\Delta(L'_t) < \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})]$. Denote by V'^B_{t+1} and V'^L_{t+1} of the continuation values for the borrower and the lender at $(t+1)^0$ following history $\{L_1, L_2, \ldots, L_{t-1}, L'_t\}$ (without default). By the fact that (l, b) is a social equilibrium, at time t with history $\{L_1, L_2, \ldots, L_{t-1}\}$ (without default), the lender should not find it profitable to offer L'_t. This implies that either L'_t does not induce default and the lender gets weakly worse off, i.e.

$$R(L'_t) + \delta V'^L_{t+1} \le R(L_t) + \delta V^L_{t+1}(\mathbf{L}), \tag{11}$$

or L'_t induces default, which means that from the borrower's viewpoint,

$$\Delta(L'_t) \ge \delta[V'^B_{t+1} - \lambda' V^B_0(\mathbf{L})]. \tag{12}$$

Since $L'_t > L_t$ and R is strictly increasing, we know that (11) implies $V'^L_{t+1} < V^L_{t+1}(\mathbf{L})$. Also, since by construction $\Delta(L'_t) < \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})]$, we know that (12) implies $V'^B_{t+1} < V^B_{t+1}(\mathbf{L})$. Therefore, either $V'^L_{t+1} < V^L_{t+1}(\mathbf{L})$ or $V'^B_{t+1} < V^B_{t+1}(\mathbf{L})$. However, note that the loan sizes in history $\{L_1, L_2, \ldots, L_{t-1}, L'_t\}$ (without default) and in $\{L_1, L_2, \ldots, L_{t-1}, L_t\}$ (without default) only differ in the last period with $L'_t > L_t$, thus we have reached a contradiction to part (ii) of Definition 2 of orthodox social equilibrium.[32] □

Lemma 3. *Let (l, b) be an orthodox social equilibrium, and $\mathbf{L} = \{L_t\}_t$ be the sequence of loan sizes on its equilibrium path. If $\{L_t\}_t$ satisfies:*

$$\Delta(L_t) = \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})], \text{ for all } t \ge 0, \tag{13}$$

where $V^B_t(\mathbf{L}) = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i})$, then $\{L_t\}_t$ is strictly monotonic or constant.

Proof. We first show that $\{L_t\}_t$ is either weakly increasing or strictly decreasing. Take any $\{L_t\}_t$ satisfying (13). From (13), we have:

$$\Delta(L_t) - \Delta(L_{t-1}) = \delta \Sigma_{\tau=t}^{\infty} \delta^{\tau-t}[C(L_{\tau+1}) - C(L_\tau)], \text{ for all } t \ge 1. \tag{14}$$

Suppose $\{L_t\}_t$ is not weakly increasing. Then there exists a t s.t. $L_{t+1} < L_t$. Since Δ and C are strictly increasing, (14) implies there must be infinitely many t, s.t. $L_{t+1} < L_t$. We consider the following 2 cases.

Case 1: There exists a T, s.t. $L_{t+1} < L_t$ for all $t \ge T$. That is, eventually $\{L_t\}_t$ becomes strictly decreasing. We claim that in this case, $\{L_t\}_t$ must be a strictly decreasing sequence (from time 0). To see this, notice that since $L_{t+1} < L_t$ for all $t \ge T$, we know from (14) that $L_T - L_{T-1} < 0$, i.e. $L_T < L_{T-1}$, because Δ and C are strictly increasing. Apply this step all the way back to $t = 1$ to obtain $L_t < L_{t-1}$ for all $t \ge 1$. So $\{L_t\}_t$ is a strictly decreasing sequence.

Case 2: There does *not* exist a T, s.t. $L_{t+1} < L_t$ for all $t \ge T$. That is, $\{L_t\}_t$ never becomes strictly decreasing after any T. Now let $t_1 \ge 1$ be the smallest time index, s.t. $L_{t_1} < L_{t_1-1}$. By the assumption in Case 2, there is a $t_2 \ge t_1$, s.t. $L_{t_2} < L_{t_2-1}$ but $L_{t_2+1} \ge L_{t_2}$. Now consider L_{t_2}. By (14), we have:

$$0 > \Delta(L_{t_2}) - \Delta(L_{t_2-1}) = \delta\{[C(L_{t_2+1}) - C(L_{t_2})] + \delta[C(L_{t_2+2}) - C(L_{t_2+1})] + \ldots\}, \tag{15}$$

$$0 \le \Delta(L_{t_2+1}) - \Delta(L_{t_2}) = \delta\{[C(L_{t_2+2}) - C(L_{t_2+1})] + \delta[C(L_{t_2+3}) - C(L_{t_2+2})] + \ldots\}. \tag{16}$$

Let:

$$\begin{aligned} K &= (1-\delta)\{[C(L_{t_2+1}) - C(L_{t_2})] + \delta[C(L_{t_2+2}) - C(L_{t_2+1})] + \ldots\} \\ &= (1-\delta)\Sigma_{\tau=t_2}^{\infty} \delta^{\tau-t_2}[C(L_{\tau+1}) - C(L_\tau)] \\ &< 0, \end{aligned} \tag{17}$$

where the inequality follows directly from (15). We claim that $(1-\delta)\{[C(L_{t_2+2}) - C(L_{t_2+1})] + \delta[C(L_{t_2+3}) - C(L_{t_2+2})] + \ldots\} < K < 0$, which is a contradiction to (16). To see this, assume that $(1-\delta)\{[C(L_{t_2+2}) - C(L_{t_2+1})] + \delta[C(L_{t_2+3}) - C(L_{t_2+2})] + \ldots\} \ge K$. Then:

$$\begin{aligned} K &= (1-\delta)\{[C(L_{t_2+1}) - C(L_{t_2})] + \delta[C(L_{t_2+2}) - C(L_{t_2+1})] + \ldots\} \\ &\ge (1-\delta)\left\{[C(L_{t_2+1}) - C(L_{t_2})] + \frac{\delta K}{1-\delta}\right\} \end{aligned}$$

[32] In fact, the contradiction is reached because the values from the rest of the game for both parties following any history (in particular, $\{L_1, L_2, \ldots, L_{t-1}, L'_t\}$ (without default)) are weakly higher than the remaining values of the current relationship following the same history (since the former includes both the latter and the values from the possibility of re-matching if the relationship is terminated at some date later).

$$
\begin{aligned}
&> (1-\delta)\left(K + \frac{\delta K}{1-\delta}\right) \\
&= K,
\end{aligned}
$$

where the second line follows from the assumption we just made and the third line follows from $C(L_{t_2+1}) - C(L_{t_2}) \geq 0 > K$. So Case 2 is not possible.

Combining Cases 1 and 2, we conclude that any $\{L_t\}_t$ satisfying (13) must be either weakly increasing or strictly decreasing.

Now we show that in the case that $\{L_t\}_t$ is weakly increasing, it must be either strictly increasing or constant. Take any $\{L_t\}_t$ satisfying (13) and weakly increasing. If it is not strictly increasing, then there exists a smallest t, call it t_3, s.t. $L_{t_3} = L_{t_3-1}$. According to (14), since $\{L_t\}_t$ is weakly increasing, RHS of (14) at $t = t_3$ is 0 only if $L_{t+1} = L_t$ for all $t \geq t_3$. Therefore. $\{L_t\}_t$ is constant from $t_3 - 1$. Now we show that $t_3 = 1$. Assume not, i.e. $t_3 \geq 2$. Since $\{L_t\}_t$ is weakly increasing and t_3 is the smallest t, s.t. $L_t = L_{t-1}$, we have $L_{t_3-1} > L_{t_3-2}$. According to (14), we should have:

$$
\Delta(L_{t_3-1}) - \Delta(L_{t_3-2}) = \delta \Sigma_{\tau=t_3-1}^{\infty} \delta^{\tau-(t_3-1)}[C(L_{\tau+1}) - C(L_\tau)]. \tag{18}
$$

Notice that RHS of (18) is 0 because we have already proved that $\{L_t\}_t$ is constant from $t_3 - 1$, whereas LHS of (18) is positive. So (18) cannot hold, a contradiction. Therefore, $t_3 = 1$, which implies that $\{L_t\}_t$ has to be a constant sequence, if it is weakly increasing but not strictly increasing.

Therefore, we conclude that for any $\{L_t\}_t$ satisfying (13), it is strictly monotonic or constant.
□

Lemma 4. *Let (l, b) be an orthodox social equilibrium, and $\mathbf{L} = \{L_t\}_t$ be the sequence of loan sizes on its equilibrium path. Then $\{L_t\}_t$ must satisfy one and only one of the following four properties:*

(i) *it is strictly increasing;*
(ii) *it is constant;*
(iii) *it is strictly decreasing;*
(iv) *it is constant at L^* until some T and then becomes strictly decreasing.*

Proof. By Lemma 2, we already know that $\{L_t\}_t$ satisfies (10). Consider the following 2 cases:

Case 1: $L_t < L^*$, for all $t \geq 0$, i.e. the 1st constraint in (10) never binds, which implies that $\{L_t\}_t$ satisfies (13). By Lemma 3, $\{L_t\}_t$ satiesfies (i), (ii) or (iii) in Lemma 4.

Case 2: $L_t = L^*$, for some $t \geq 0$. Note first that if $L_T = L^*$, then $L_{T-1} = L^*$, which implies $L_t = L^*$ for all $t \leq T$. To see this, consider time $T - 1$. The RHS of the 2nd constraint in (10) at time $T - 1$ is: $\delta[V_T^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$. Notice that:

$$
\begin{aligned}
V_T^B(\mathbf{L}) &= \Sigma_{i=0}^{\infty} \delta^i C(L_{T+i}) \\
&= C(L_T) + \delta V_{T+1}^B(\mathbf{L}) \\
&= C(L^*) + \delta V_{T+1}^B(\mathbf{L}) \\
&\geq V_{T+1}^B(\mathbf{L}),
\end{aligned}
$$

where the last line follows from $V_{T+1}^B(\mathbf{L}) \leq \frac{C(L^*)}{1-\delta}$ because C is strictly increasing. Then we have:

$$
\delta[V_T^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \delta[V_{T+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \Delta(L^*). \tag{19}
$$

(19) implies that at time $T - 1$, L^* is the solution to (10). Apply this argument all the way back to $t = 0$ to obtain that $L_t = L^*$ for all $t \leq T$.

Now let t_4 be the period s.t. $L_t = L^*$ for all $t \leq t_4$, and $L_t < L^*$ for all $t \geq t_4 + 1$. If such a t_4 does not exist, then $\{L_t\}_t$ is a constant sequence at L^*, which satisfies (ii) in Lemma 4, so we're done. If t_4 exists, we claim that $L_{t+1} > L_t$, for all $t \geq t_4$.

To see this, notice first that the 2nd constraint in (10) is binding for all $t \geq t_4 + 1$ because $L_t < L^*$ for all these t's. Then by Lemma 3, $\{L_t\}_t$ from $t = t_4 + 1$ has to be strictly increasing, or strictly decreasing or constant. We now show that it must be strictly decreasing. Assume not, then it must be strictly increasing or constant from $t_4 + 1$. By definition of $V_t^B(\mathbf{L})$, we have $V_{t_4+2}^B(\mathbf{L}) \geq V_{t_4+1}^B(\mathbf{L})$, which implies:

$$
\delta[V_{t_4+2}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \delta[V_{t_4+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \Delta(L^*), \tag{20}
$$

where the last inequality follows from $L_{t_4} = L^*$. But this implies that at time $t_4 + 1$, L^* is the solution to (10), a contradiction to $\{L_t\}_t$ being on equilibrium path and $L_{t_4+1} < L^*$. Therefore, $\{L_t\}_t$ has to be strictly decreasing from $t_4 + 1$, meaning that it satisfies (iv) in Lemma 4. □

Proof of Proposition 1. Let $\bar{\alpha} = \sup_{L \in (0, L^*]} \frac{C(L)}{D(L)}$ and $\underline{\alpha} = \inf_{L \in (0, L^*]} \frac{C(L)}{D(L)}$. Under Assumption 1, we have $0 < \underline{\alpha} \leq \bar{\alpha} < 1$. Now define $\delta^* \equiv 1 - \underline{\alpha}$ and $\lambda_\delta^* \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$. We will show that such δ^* and λ_δ^* work.

As we already defined, $\lambda' = \frac{\lambda}{1-(1-\lambda)\delta}$. It can be checked that $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, iff $\lambda > \frac{\bar{\alpha}+\delta-1}{\delta}$.

Call a sequence $\{L_t\}_t$ an equilibrium loan sequence if it satisfies (10) for all t. We first that show any non-trivial equilibrium loan sequence $\{L_t\}_t$ cannot be a constant sequence. Assume it were, i.e. $0 < L_t = \tilde{L} \le L^*$ for all $t \ge 0$. Then we directly know that $V_t^B(\mathbf{L}) = \frac{C(\tilde{L})}{1-\delta}$ for all t. Now we check the 2nd constraint in (10):

$$\begin{aligned}\delta[V_t^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] &= \delta\left[\frac{C(\tilde{L})}{1-\delta} - \lambda' \frac{C(\tilde{L})}{1-\delta}\right]\\ &< \delta\left(1 - \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}\right)\frac{C(\tilde{L})}{1-\delta}\\ &= \delta\frac{(1-\bar{\alpha})(1-\delta)}{\delta\bar{\alpha}}\frac{C(\tilde{L})}{1-\delta}\\ &= \frac{(1-\bar{\alpha})}{\bar{\alpha}}C(\tilde{L})\\ &\le D(\tilde{L}) - C(\tilde{L}),\end{aligned}$$

where the second line (strictly inequality) follows from the condition $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, and the last line follows from $\bar{\alpha} = \sup_{L\in[0,L^*]}\frac{C(L)}{D(L)} \ge \frac{C(\tilde{L})}{D(\tilde{L})}$. This implies that (10) is not satisfied (at any t), a contradiction to $\{L_t\}_t$ being an equilibrium loan sequence. Therefore $\{L_t\}_t$ cannot be a constant sequence when $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$.

By Lemma 4 and the result above, we know that $\{L_t\}_t$ is convergent, and (13) is eventually satisfied by $\{L_t\}_t$ after some T; that is, there exists a T, s.t. (13) is satisfied for all $t \ge T$.[33] Let $\bar{L}$ be the limit of $\{L_t\}_t$. By definition of $V_t^B(\mathbf{L})$, it converges to $\frac{C(\bar{L})}{1-\delta}$. As (13) is eventually satisfied, we must have:

$$D(\bar{L}) - C(\bar{L}) = \delta[\frac{C(\bar{L})}{1-\delta} - \lambda' V_0^B(\mathbf{L})], \tag{21}$$

which gives us:

$$\begin{aligned}V_0^B(\mathbf{L}) &= \frac{\delta C(\bar{L}) - (1-\delta)[D(\bar{L}) - C(\bar{L})]}{(1-\delta)\delta\lambda'}\\ &\le \frac{\delta - (1-\delta)(\frac{1}{\bar{\alpha}} - 1)}{(1-\delta)\delta\lambda'}C(\bar{L})\\ &< \frac{C(\bar{L})}{1-\delta},\end{aligned} \tag{22}$$

where the second line follows from $\frac{D(\bar{L})}{C(\bar{L})} \ge \frac{1}{\bar{\alpha}}$, and the third line is obtained by using $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$.

Among the four possible properties in Lemma 4 one of which $\{L_t\}_t$ has to satisfy, only if $\{L_t\}_t$ is strictly increasing will (22) hold. Therefore we conclude that when $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, the loan sizes along the equilibrium path of any non-trivial orthodox social equilibrium must be strictly increasing. $\square$

Proof of Proposition 2. We keep the definition of δ^* and λ^*_δ, i.e. $\delta^* \equiv 1 - \underline{\alpha}$ and $\lambda^*_\delta \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$, where $\bar{\alpha} = \sup_{L\in(0,L^*]}\frac{C(L)}{D(L)}$ and $\underline{\alpha} = \inf_{L\in(0,L^*]}\frac{C(L)}{D(L)}$.

We first establish the existence of an equilibrium loan sequence, i.e. the sequence that satisfies (10). Define:

$$\hat{D}(L) = \begin{cases} D(L), & \text{if } L \in [0, L^*];\\ D(L^*) + (L - L^*), & \text{if } L > L^*;\end{cases}$$
$$\hat{C}(L) = \begin{cases} C(L), & \text{if } L \in [0, L^*];\\ C(L^*) + \frac{C(L^*)}{D(L^*)}(L - L^*), & \text{if } L > L^*.\end{cases}$$

Notice that by construction, $\hat{D}$ and $\hat{C}$ are continuous extensions of D and C, respectively; both are strictly increasing and unbounded above, with $\inf_{L>0}\frac{\hat{C}(L)}{\hat{D}(L)} = \inf_{L\in[0,L^*]}\frac{C(L)}{D(L)} = \underline{\alpha}$ and $\sup_{L>0}\frac{\hat{C}(L)}{\hat{D}(L)} = \sup_{L\in[0,L^*]}\frac{C(L)}{D(L)} = \bar{\alpha}$.

As implied by Lemma 4, any equilibrium loan sequence converges. Given this property, pick any $\bar{L} \in [0, L^*]$, we can construct as follows a unique equilibrium loan sequence $\{L_t\}_t$ converging to $\bar{L}$:

$$V_0^B = \frac{\frac{C(\bar{L})}{1-\delta} - D(\bar{L})}{\delta\lambda'}, \tag{23}$$

[33] This is because, the only case where (13) is not necessarily eventually satisfied (i.e. the 2nd constraint of (10) is not eventually binding) is that $\{L_t\}_t$ is constant at L^*, which has just been ruled out.

$$L_t = \hat{D}^{-1}(V_t^B - \delta\lambda' V_0^B), \tag{24}$$

$$V_{t+1}^B = \frac{V_t^B - \hat{C}(L_t)}{\delta}. \tag{25}$$

We first show that $\{L_t, V_t^B\}_t$ constructed above is well-defined. It is enough to show that $V_t^B - \delta\lambda' V_0^B > 0$ for all t because then by the fact that $\hat{D}$ is continuous, strictly increasing and unbounded we know L_t is well-defined. To see $V_t^B - \delta\lambda' V_0^B > 0$, note that (24) implies that $\{V_t^B\}_t$ satisfies:

$$\hat{D}(L_t) + \delta\lambda' V_0^B = V_t^B.$$

Combining with (25) we have:

$$\frac{\hat{D}(L_t)}{\hat{C}(L_t)}(V_t^B - \delta V_{t+1}^B) + \delta\lambda' V_0^B = V_t^B.$$

Rearranging, we get:

$$V_{t+1}^B = \frac{1 - \frac{\hat{C}(L_t)}{\hat{D}(L_t)}}{\delta} V_t^B + \frac{\hat{C}(L_t)}{\hat{D}(L_t)}\lambda' V_0^B. \tag{26}$$

By (23) and the assumption that $\delta \geq 1 - \inf \frac{C(L)}{D(L)}$, we first have $V_0^B > 0$, so that $V_0^B - \lambda'\delta V_0^B > 0$. Then by induction on (26), we have $V_t^B > \delta\lambda' V_0^B$ for all t. Therefore $\{L_t, V_t^B\}_t$ constructed in (23) through (25) is well-defined.

Now we show that $\{V_t^B\}_t$ is bounded. Consider another sequence $\{\hat{V}_t\}_t$ s.t. $\hat{V}_0 = V_0^B$ and

$$\hat{V}_{t+1} = \frac{1-\underline{\alpha}}{\delta}\hat{V}_t + \hat{V}_0. \tag{27}$$

Since for all t, $1 > \frac{\hat{C}(L_t)}{\hat{D}(L_t)} \geq \underline{\alpha} \equiv \inf \frac{\hat{C}(L)}{\hat{D}(L)}$, we know from (26) and (27) that $V_t^B \leq \hat{V}_t$ for all t. Note also that the solution to $\{\hat{V}_t\}_t$ is:

$$\hat{V}_t = \hat{V}_0\left(\frac{\delta}{\underline{\alpha} + \delta - 1} - \frac{1-\underline{\alpha}}{\underline{\alpha} + \delta - 1}(\frac{1-\underline{\alpha}}{\delta})^t\right), \tag{28}$$

which is bounded because $\frac{1-\underline{\alpha}}{\delta} < 1$ as assumed. Then we know that $\{V_t^B\}_t$ is bounded above, and because $V_t^B - \delta\lambda' V_0^B > 0$ for all t and $V_0^B > 0$ as we have shown, it is also bounded below. Thus $\{V_t^B\}_t$ is bounded.

Now we claim that $\{L_t, V_t^B\}_t$ satisfies: for all $t \geq 0$,

$$\hat{D}(L_t) - \hat{C}(L_t) = \delta[V_{t+1}^B - \lambda' V_0^B], \tag{29}$$

$$V_t^B = \Sigma_{i=0}^{\infty}\delta^i \hat{C}(L_{t+i}). \tag{30}$$

It can be checked that (29) is obtained by substituting (25) into (24), and (30) is obtained by expanding (25) recursively and using the boundedness of $\{V_t^B\}_t$.

By applying Lemma 3 to (29) and (30),[34] we know that $\{L_t\}_t$ is monotonic. Since $\hat{C}$ are strictly increasing, V_t^B by (30) is monotonic. Because $\{V_t^B\}_t$ is also bounded, as we have just shown, $\{V_t^B\}_t$ is convergent. Then by (29) again, $\{L_t\}_t$ is also convergent. But then, by construction of V_0^B in (23), we have $L_t \to \bar{L}$.

Because $\{L_t\}_t$ can only be strictly increasing, strictly decreasing or constant (by Lemma 3), then using the condition $\lambda > \lambda_{\delta}^* \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$ and by the same deduction as in (22), we know that L_t strictly increases to $\bar{L} \leq L^*$. This in turn implies that $L_t \in [0, L^*]$ for all t, and by definition of $\hat{D}$ and $\hat{C}$ as well as (29) and (30), we have:

$$\Delta(L_t) = \delta[V_{t+1}^B - \lambda' V_0^B], \tag{31}$$

$$V_t^B = \Sigma_{i=0}^{\infty}\delta^i C(L_{t+i}). \tag{32}$$

Therefore, $\{L_t\}_t$ satisfies (10), meaning that it is an equilibrium loan sequence converging to $\bar{L}$.

[34] Note that the proof of Lemma 3 only use the conditions that D, C and $D - C$ are strictly increasing, which hold for $\hat{D}$, $\hat{C}$ and $\hat{D} - \hat{C}$ here.

Finally we show the existence of an orthodox social equilibrium by construction. Let $\mathbf{L}^* = \{L_t^*\}_t$ be the (unique) equilibrium loan sequence converging to $\bar{L} = L^*$. We construct (l, b) as follows: $l = \{l_0, l_1, \ldots\}$, $b = \{b_0, b_1, \ldots\}$, where for all t, $l_t = (\tilde{L}_{t^0}, \tilde{f}_{t^2})$, $b_t = (\tilde{d}_{t^1}, \tilde{g}_{t^2})$, in which:

$$\tilde{L}_{t^0}[h(t^0)] = \begin{cases} L_t^*, & \text{if } d_\tau = 0, \text{ for all } \tau < t; \\ 0, & \text{otherwise}; \end{cases} \tag{33}$$

$$\tilde{d}_{t^1}[h(t^1)] = \begin{cases} 0, & \text{if } \Delta(L_t) \le \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]; \text{ and for all } \tau < t, d_\tau = 0; \\ 1, & \text{otherwise}; \end{cases} \tag{34}$$

$$\tilde{f}_{t^2}[h(t^2)] = \begin{cases} 1, & \text{if for all } \tau \le t, d_\tau = 0; \\ 0, & \text{otherwise}; \end{cases} \tag{35}$$

$$\tilde{g}_{t^2}[h(t^2)] = \begin{cases} 1, & \text{if for all } \tau \le t, d_\tau = 0; \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Recall that $\tilde{L}_{t0}$ is the loan offered in period t, $\tilde{d}_{t1}$ is the defaulting decision in period t, and $\tilde{f}_{t2}$ and $\tilde{g}_{t2}$ are continuation decisions in period t. Eq. (33) through (36) describe the following strategy profile.

- For the lender, in period t, as long as default has never happened, she offers L_t^* and continues the relationship, *regardless of whether or not they are on the equilibrium path* (i.e. regardless of previous loan sizes)[35]; if default has happened before, she offers 0 loan and terminates the relationship.
- For the borrower, in period t, she will repay as long as the current benefit from defaulting is less than its future cost and default has never happened; otherwise, she will default. The borrower continues the relationship if and only if she never defaulted before.

One particular feature of this construction is that, no matter L_t is on or off equilibrium path, the borrower's perception of her future value is always the one on path, i.e. $V_{t+1}^B(\mathbf{L}^*)$, as long as default never happened. This is a correct perception given the lender's strategy.

Now we show that the constructed (l, b) is an orthodox social equilibrium. By construction of $\tilde{L}_{t0}$ in (33), part (ii) of Definition 2 is satisfied because at the beginning of a given date the continuation values are always the same as long as there is no default. By construction of $\tilde{f}_{t2}$ in (35), part (i) of Definition 2 are satisfied. It remains to be shown that (l, b) is a social equilibrium, i.e. given that the re-match values being V_0^L and V_0^B, (l, b) are sequentially rational with respect to each other. Since this is a game with complete information, it is sufficient to check for one-shot deviation at all possible histories.

First consider $h(t^0)$. At history $h(t^0)$ s.t. $d_\tau = 0$ for all $\tau < t$, i.e. there is no default before t, if the lender sets $L_t > L_t^*$, we know from (34) that this will induce default. But then by (35), this will result in the termination of the current relationship. So by setting $L_t > L_t^*$, the lender's payoff changes from $V_t^L(\mathbf{L}^*)$ to $\delta\lambda' V_0^L(\mathbf{L}^*)$, where $V_t^L(\mathbf{L}^*) = \Sigma_{i=0}^\infty \delta^i R(L_{t+i})$. As $\{L_t^*\}_t$ is strictly increasing, $V_t^L(\mathbf{L}^*) > V_0^L(\mathbf{L}^*) > \delta\lambda' V_0^L(\mathbf{L}^*)$, so this deviation is not profitable. If the lender sets $L_t < L_t^*$, based on (l, b) the borrower will not default and the loan sequence $\{L_t^*\}_t$ will be restored from next period; so this deviation just lowers the lender's current payoff from $R(L_t^*)$ to $R(L_t)$ while keeping future payoff constant, which is not profitable.

At history $h(t^0)$ s.t. $d_\tau = 1$ for some $\tau < t$, i.e. there exists default before t but the relationship is not yet terminated, we know from (34) and (35) that no matter what the lender offers, the borrower will default and the relationship will be terminated at the end of this period. Then offering anything larger than 0 will not be a profitable deviation for the lender.

Now consider $h(t^1)$. At history $h(t^1)$ s.t. $\Delta(L_t) \le \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$ and $d_\tau = 0$, for all $\tau < t$, a one-shot deviation to default will result in the termination of the current relationship, which is not profitable for the borrower exactly because $\Delta(L_t) \le \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$.

At history $h(t^1)$ s.t. $\Delta(L_t) > \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$ or $d_\tau = 1$ for some $\tau < t$, i.e. either the extra payoff from default is higher than its cost, or there exists default record before t (or both) so that the current relationship will be terminated at the end of this period no matter whether she defaults. In both cases the borrower is better off by defaulting. So there is no profitable deviation.

Finally consider $h(t^2)$. At history $h(t^2)$ s.t. $d_\tau = 0$ for all $\tau \le t$, i.e. there is no default at or before t, if the lender one-shot deviates to terminating the relationship, she will get $R(L_t^*) + \delta\lambda' V_0^L(\mathbf{L}^*)$ instead of $R(L_t^*) + \delta V_{t+1}^L(\mathbf{L}^*)$, which is not profitable because $\{V_t^L(\mathbf{L}^*)\}_t$ is increasing in t. Similarly, if the borrower one-shot deviates to terminating the relationship, she will get $C(L_t^*) + \delta\lambda' V_0^B(\mathbf{L}^*)$ instead of $C(L_t^*) + \delta V_{t+1}^B(\mathbf{L}^*)$, which is not profitable because $\{V_t^B(\mathbf{L}^*)\}_t$ is increasing in t.

At history $h(t^2)$ s.t. $d_\tau = 1$ for some $\tau < t$, i.e. there exists default at or before t but the relationship is not yet terminated, if the lender or the borrower one-shot deviates to continuing the relationship, the relationship will still be terminated

[35] In other words, as long as default never occurred, the lender always offers the on-path loan size L_t^* even if the previous loans were off equilibrium path.

because the other party will do so according to her equilibrium strategy. So such a one-shot deviation will not change anything, which is not profitable.

Therefore, at no history can we find a profitable one-shot deviation for any player, so (l, b) are sequentially rational with respect to each other.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2018.11.025.

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