ORA E. PERCUS AND JEROME K. PERCUS

Can Two Wrongs Make a Right? Coin-Tossing Games and Parrondo's Paradox

A number of natural and man-made activities can be cast in the form of various one-person games, and many of these appear as sequences of transitions without memory, or Markov chains. It has been observed, initially with surprise, that losing games can often be combined by selection, or even randomly, to result in winning games. Here, we present the analysis of such questions in concise mathematical form (exemplified by one nearly trivial case and one which has received a fair amount of prior study), showing that two wrongs can indeed make a right—but also that two rights can make a wrong!

Background

On frequent occasions, a logical oddity comes along, which attracts a sizeable audience. One of the most recent is known as Parrondo's paradox [5, 6]. Briefly, it is the observation that random selection (or merely alternation) of the playing of two asymptotically losing games* can result in a winning game.

Conceptually similar situations involving only the processing of statistical data are not novel. What has been referred to as Simpson's paradox [8] is typified by this scenario: Quite different items, say type 1 and type 2, cost dealers the same \$10 per unit. Suppose that, during a given period, dealer A sells 20 and 80 of these two types, charging \$13 and \$15, respectively, per item. Dealer B, on the other hand, who charges \$14 and \$16 per item, sells 80 and 20 of the two types. Then the average cost per item to dealer A's customers is (1/5)13 + (4/5)15 = \$14.60, while B's on the average only pay (4/5)14 + (1/5)16 = \$14.40, a net result that B is delighted to advertise. This despite the fact that A sells both items more cheaply than B does! No surprise, since A sells mainly the more expensively marked

*Such a game consists of repeated moves where the expected net gain per move is negative.

item and B the cheaper one; this kind of cheating with statistics must be commonplace.

In fact, it has been pointed out by Saari [7] that aggregation often yields statistical results qualitatively different from those apparent at a lower level, and that this is related to well-known problems in game theory and economic theory.

Still, turning two losing games into a winning one (now we are playing solitaire) seems more than a bit counterintuitive. To demystify it a little, consider a really extreme case of the Parrondo phenomenon in which in game A, a player can only move from white to black, or black to black. The outcome of a single move is a gain of \$3 if the player moves from white to black, and a loss of \$1 each time he moves from black to black. Because the player becomes trapped on the losing color, his expected gain per move is $-\$1$. In game B, the role of black and white are reversed, but the expected gain per move is the same $-\$1$. Now a random selection of game A or game B results in an expected gain of \$1 per move, no matter what color one is moving from (because half the time, whether A or B is played, you are moving from a winning color and half the time from a losing color). What is happening is that each game is rescuing the other.

Examples that are given need not be so obvious (we will quote a prototype later), and it is worthwhile having a mathematical structure to organize their analysis. If the "game" concept is restricted sufficiently to allow a clear interpretation of the averaging strategy mentioned above, this is readily accomplished.

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The Expected Gain

Let's get technical! By a (one-person) game, we will mean a set of transitions from state j (among a finite set of states S of size s) to state i, with transition probability T_{ij}; in addition, to the move $j \to i$ in this Markov chain [9] we must associate a gain w_{ij}, which can be positive or negative. Of course, $T_{ij} \geq 0$, and $\sum_{i \in S} T_{ij} = 1$ for any $j \in S$, which can be written in vector-matrix form as

$$1^t T = 1^t, \tag{3.1}$$

where 1 is the column vector of all 1's, and superscript t indicates transpose. The properties of such stochastic matrices are an old story, and in particular, we will confine our attention to the large class of irreducible stochastic matrices, where if one starts with a probability vector $p_{0j} = Pr(\text{start in state } j)$ for the possible states, then iteration of the process

$$p_0, Tp_0, T^2 p_0, \ldots$$

results asymptotically in the unique mix of state probabilities $\phi_{0,j}$ for state j, regarded as components of the probability vector $\{\phi_{0,j}\}$ satisfying

$$T\phi_0 = \phi_0. \tag{3.2}$$

This corresponds to the eigenvalue $\lambda_0 = 1$ of the matrix T, all other eigenvalues being simple and having smaller absolute values.

Now let us combine the matrix T and the set of gains $\{w_{ij}\}$ to form the matrix $T(x)$, defined by

$$T_{ij}(x) = T_{ij}\, x^{w_{ij}}, \tag{3.3}$$

i.e., introduce a weight for the $j \to i$ transition of x raised to the w_{ij} power. The reason for doing so is that if we consider any sequence of transitions $j_0, j_1, \ldots j_N$ from an initial j_0, then this sequence has a probability $Pr(j_0, \ldots j_N) = T_{j_N j_{N-1}} \cdots T_{j_2 j_1} T_{j_1 j_0}$, and an associated gain $W_N(j_0, \ldots j_N) = w_{j_N j_{N-1}} + \cdots + w_{j_1 j_0}$, so that

$$Pr(j_0, \ldots j_N) x^{W_N(j_0, \ldots j_N)} = \prod_{n=1}^{N} T_{j_n j_{n-1}}(x). \tag{3.4}$$

By summing over all N-step sequences, we produce the powerful moment-generating function of W_N, given by the expectation

$$E(x^{W_N}) = 1^t T(x)^N p_0. \tag{3.5}$$

The moment-generating function is a wonderful tool for finding expectation values, and we'll use it right away. To do so, we first have to get a handle on $T(x)^N$. Suppose that $\lambda(x)$ is the maximum eigenvalue of $T(x)$; if x is real and close to 1, $\lambda(x)$ will still be real, close to 1, and largest in absolute value. Furthermore, if we normalize the maximal right eigenvector $\phi_0(x)$ of $T(x)$ by $1^t\phi_0(x) = 1$, and the corresponding left eigenvector $\psi_0(x)$ of $T(x)$ by $\psi_0(x)^t\ \phi_0(x) = 1$, then $T(x)^N/\lambda(x)^N$ approaches the corresponding projection:

$$\lim_{N\to\infty} T(x)^N/\lambda(x)^N = \phi_0(x)\ \psi_0^t(x). \tag{3.6}$$

Hence (3.5) implies that

$$\lim_{N\to\infty} E(x^{W_N})/\lambda(x)^N = \psi_0^t(x) p_0. \tag{3.7}$$

There is a lot of information in (3.7), but we will concentrate on the asymptotic gain per move,

$$\overline{w} = \lim_{N\to\infty} E(W_N)/N. \tag{3.8}$$

To find it, just differentiate (3.7) with respect to x and set $x = 1$, assuming commutativity of the limiting operations. Because $\lambda(1) = 1$, $\phi_0(1) = \phi_0$, $\psi_0(1) = 1$, we have $\lim_{N\to\infty} (E(W_N) - N\lambda'(1)) = \psi_0'^t(1)p_0$, which is finite. Hence $\lim_{N\to\infty} \frac{1}{N}(E(W_N) - N\lambda'(1)) = 0$, or according to (3.8)

$$\overline{w} = \lambda'(1). \tag{3.9}$$

An even more transparent alternative representation is obtained by differentiating $T(x)\phi_0(x) = \lambda(x)\phi_0(x)$ with respect to x and setting $x = 1$: $T'(1)\phi_0 + T\phi_0'(1) = \lambda'(1)\phi_0 + \phi_0'(1)$. Taking the scalar product with 1:

$$1^t T'(1)\phi_0 + 1^t\phi_0'(1) = \lambda'(1) + 1^t\phi_0'(1),$$

so that $\lambda'(1) = 1^t T'(1)\phi_0$. Thus,

$$\overline{w} = 1^t T'(1)\phi_0, \qquad (3.10)$$

whose interpretation is obvious: ϕ_0 is the asymptotic state vector whose components are $\phi_{0,k}$, $k = 1, \ldots s$; $T'_{ij}(1) = w_{ij}T_{ij}$ is the gain per move weighted by its probability; and 1^t adds it all up. Hence

$$w_k = \sum_i w_{ik}T_{ik} \qquad (3.11)$$

is the expected gain on making a move from state k, and we can also write (3.10) in the form

$$\overline{w} = \sum_k w_k\phi_{0,k}. \qquad (3.12)$$

Game Averaging—a Simple Example

A *game*, in the terminology we have been using, is fully specified by the weighted transition matrix $T(x)$, which tells us at the same time the probability T_{ij} of a transition $j \to i$ and the gain w_{ij} produced by that move. A random composite of games A and B can then be created by choosing, prior to each move, which game is to be played; A (and its associated move probability and gain per move), say, with probability a; or B, with probability $1 - a$.

$$T_{A,B}(x) = aT_A(x) + (1 - a)T_B(x). \qquad (4.1)$$

What has come to be known as Parrondo's paradox (originally, a rough model of the "flashing ratchet" [1]), is that domain in which both $\overline{w}_A < 0$ and $\overline{w}_B < 0$, but $\overline{w}_{A,B} > 0$. Much of the phenomenology is already present in a variant of the simple model we have mentioned as background. Let us see how this goes:

In both games, A and B, a move is made from white or black to white or black. Game A is now defined by a probability p, no longer unity, of moving *to* black, $q = 1 - p$ *to* white, with a gain of \$3 on a move *from* white, of $-\$1$ on a move *from* black. Hence (with white : $j = 1$, black : $j = 2$)

$$T_A = \begin{pmatrix} q & q \\ p & p \end{pmatrix}, \qquad \phi_{0A} = \begin{pmatrix} q \\ p \end{pmatrix},$$
$$T_A(x) = \begin{pmatrix} qx^3 & q/x \\ px^3 & p/x \end{pmatrix}; \qquad (4.2)$$

in game B, the roles of black and white are reversed, so that

$$T_B = \begin{pmatrix} p & p \\ q & q \end{pmatrix}, \qquad \phi_{0B} = \begin{pmatrix} p \\ q \end{pmatrix},$$
$$T_B(x) = \begin{pmatrix} p/x & px^3 \\ q/x & qx^3 \end{pmatrix}. \qquad (4.3)$$

For the composite game, we imagine equal probabilities, $a = \frac{1}{2}$, of choosing one game or the other, and indicate this by $\frac{1}{2}A + \frac{1}{2}B$, and now

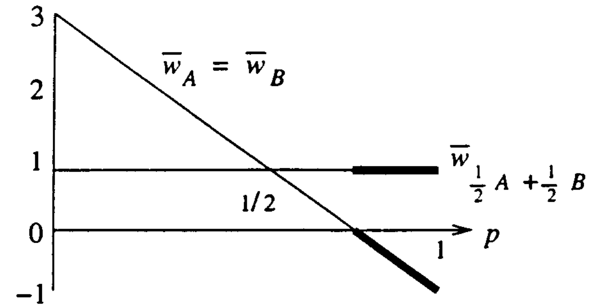

FIGURE 1.

$$T_{\frac{1}{2}A+\frac{1}{2}B} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \phi_{0,\frac{1}{2}A+\frac{1}{2}B} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$
$$w_{\frac{1}{2}A+\frac{1}{2}B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \qquad (4.4)$$

It follows, most directly from (3.10), that

$$\overline{w}_A = \overline{w}_B = 3 - 4p,\ \overline{w}_{\frac{1}{2}A+\frac{1}{2}B} = 1. \qquad (4.5)$$

Hence, in the bold region of Figure 1, for $3/4 < p \le 1$, we indeed have $\overline{w}_A = \overline{w}_B < 0$, together with $\overline{w}_{\frac{1}{2}A+\frac{1}{2}B} > 0$. (Note however that $\overline{w}_A = \overline{w}_B > \overline{w}_{\frac{1}{2}A+\frac{1}{2}B}$ for $p < \frac{1}{2}$.)

Game Averaging—Another Example

The game originally quoted in this context is as follows [2]: Each move results in a gain of $+1$ or -1 in the player's capital. If the current capital is not a multiple of 3, coin I is tossed, with a probability p_1 of winning $+1$, a probability $q_1 = 1 - p_1$ of "winning" -1. If the capital is a multiple of 3, one instead flips coin II with corresponding p_2 and q_2. Hence the states can be taken as $(-1, 0, 1)$ (mod 3), and the associated transition and gain matrices are

$$T = \begin{pmatrix} 0 & q_2 & p_1 \\ p_1 & 0 & q_1 \\ q_1 & p_2 & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \qquad (5.1)$$

The asymptotic state, satisfying $T\phi_0 = \phi_0$, is readily found:

$$\phi_0 = \begin{pmatrix} q_2 + p_1p_2 \\ 1 - p_1q_1 \\ p_2 + q_1q_2 \end{pmatrix} \Big/ (2 + p_1p_2 + q_1q_2 - p_1q_1), \qquad (5.2)$$

and then

$$\overline{w} = 1^t T'(1)\phi_0 = 3\frac{p_1^2p_2 - q_1^2q_2}{2 + p_1p_2 + q_1q_2 - p_1q_1}. \qquad (5.3)$$

Now suppose there are two games, the second specified by parameters p_1', q_1', p_2', q_2'. An averaging of the two would then define a move as: (1) choose game No. 1—call it A—with probability a, game No. 2, B with probability $1 - a$; (2) play the game chosen. Because the gain matrix w is the same for both games, this is completely equivalent to playing a new game with parameters $\hat{p}_1 = ap_1 + (1 - a)p_1'$, $\hat{p}_2 = ap_2 + (1 - a)p_2'$, etc., and so (5.3) applies as well. The "paradox" is most clearly discerned by imagining both games as fair, i.e., $p_1^2p_2 = q_1^2q_2$, or equivalently

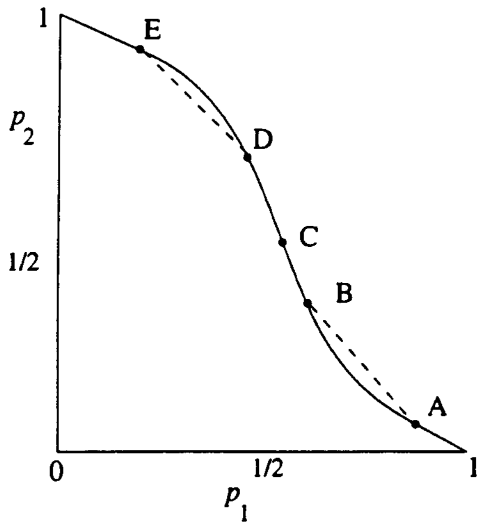

FIGURE 2.

$$p_2 = q_1^2/(p_1^2 + q_1^2), \tag{5.4}$$

and similarly for p_1', p_2', creating the "operating curve" shown in Figure 2; winning games are above the curve; losing games, below. For games A and B as marked, all averaged games lie on the dotted line between A and B, and all are winning games. And by continuity with respect to all parameters, it is clear that if A and B were slightly losing, most of the connecting dotted line would still be in the winning region. However, two slightly winning games, close to D and E, would result mainly in a losing game. So much for the paradox!

The example most frequently quoted is specialized in that game B has only one coin, equivalent to two identical coins, $p_1' = p_2'$ (= 1/2 for a fair game, point C); and is modified in that A and B are systematically switched, rather than randomly switched. Qualitatively, this is much the same.

Asymptotic Variance

Much of the activity that we have been discussing arose from extensive computer simulations [3, 4], carried out to the point of negligible fluctuations in the gain. How far does one have to go to accomplish this? A standard criterion involves looking at the variance of the gain per move as a function of the number of moves, N, that have been made:

$$\sigma^2(w; N) = E((W_N/N)^2) - (E(W_N/N))^2. \tag{6.1}$$

The computation of $\sigma^2(w; N)$ proceeds routinely from the same starting point (3.7) used previously to compute $\overline{w} = \lim_{N\to\infty} E(W_N/N)$. This time, differentiate (3.7) both once and twice with respect to x and set $x = 1$, again assuming commutativity of limiting operations. Again using $\lambda(1) = 1$, $\phi_0(1) = \phi_0$, $\psi_0(1) = 1$, this results in

$$\begin{aligned} &\lim_{N\to\infty} (E(W_N) - N\,\lambda'(1)) = \psi_0'^t(1)p_0 \\ &\lim_{N\to\infty} [E(W_N(W_N - 1)) - 2N\,E(W_N)\lambda'(1) - N\lambda''(1) \\ &\quad + N(N-1)\,\lambda'(1)^2] = \psi_0''^t(1)p_0, \end{aligned} \tag{6.2}$$

which we combine to read

$$\begin{aligned} \lim_{N\to\infty} [E(W_N^2) - (E(W_N))^2 - N\,\lambda''(1) - N\,\lambda'(1)^2 - N\,\lambda'(1)] \\ = \psi_0''^t(1)p_0 - (\psi_0'^t(1)p_0)^2 + \psi_0'^t(1)p_0. \end{aligned} \tag{6.3}$$

We see then that

$$\lim_{N\to\infty} N\,\sigma^2(w; N) = \lambda''(1) + \lambda'(1)^2 + \lambda'(1). \tag{6.4}$$

In other words, we have found that the standard deviation is given asymptotically in N by

$$\sigma(w; N) \to N^{-1/2}[\lambda''(1) + \lambda'(1)^2 + \lambda'(1)]^{1/2}, \tag{6.5}$$

with a readily computable coefficient. For example, in the "Parrondo" case of (5.1), where

$$T(x) = \begin{pmatrix} 0 & q_2/x & p_1x \\ p_1x & 0 & q_1/x \\ q_1/x & p_2x & 0 \end{pmatrix}, \tag{6.6}$$

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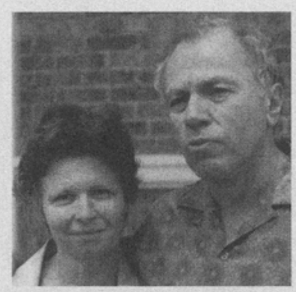

ORA E. PERCUS **JEROME K. PERCUS**
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, NY 10012 USA

Ora E. Percus received an M.Sc. in Mathematics at Hebrew University, Jerusalem, and a Ph.D. in Mathematical Statistics from Columbia University in 1965. She has been active in several areas of mathematics, including probability, statistics, and combinatorics.

Jerome K. Percus received a B.S. in Electrical Engineering, an M.A. in Mathematics, and a Ph.D. in Physics, in 1954, from Columbia University. He has worked in numerous areas of applied mathematics, primarily in chemical physics, mathematical biology, and medical statistics.

They have had many collaborations, but the best of them cannot be found in the scientific literature under their names; instead, they are called Orin and Allon.

we find that $\lambda(x)$ satisfies

$$\lambda(x)^3 - (p_1q_2 + q_1p_2 + p_1q_1)\lambda(x) + q_1^2q_2/x + p_1^2p_2x = 0. \quad (6.7)$$

By successive differentiation with respect to x, followed by $x = 1$, it follows that

$$\lambda'(1) = (p_1^2p_2 - q_1^2q_2)/D$$
$$\lambda''(1) = (-2\lambda'(1) + 2p_1^2p_2 + 4q_1^2q_2)/D \quad (6.8)$$
$$\text{where } D = \frac{1}{3}(2 + p_1p_2 + q_1q_2 - p_1q_1),$$

and so we have

$$N^{1/2}\sigma(w; N) \rightarrow \frac{2q_1}{D}[q_2(1 + p_1^2p_2)]^{1/2}. \quad (6.9)$$

Concluding Remarks

We have shown here that Parrondo's "paradox" operates in two regions. One can win at two losing games by switching between them, but one can also lose by switching between two winning games. The precise fashion in which these occur of course depends upon details of the games involved. Aside from details, the take-home message is that the procedure of averaging strategies to improve the outcome—in essence allowing each one to rescue the other—is effective under a large variety of circumstances. It is certainly taken advantage of by nature and man, although not necessarily in the transparent form of the discussion of equation (5.4).

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Puzzle Solution for Cross-Number Puzzle (24, no. 2, p. 76)

3	1	4	1	5	9	2	6	5	3
1	0	4	0	■	3	3	3	3	■
6	1	■	1	0	3	5	4	8	7
2	0	■	■	0	1	■	2	8	9
2	■	7	4	7	■	1	6	1	2
7	8	5	5	■	5	6	3	■	3
7	0	8	■	1	1	■	■	0	5
6	8	2	0	0	2	9	■	2	1
■	4	4	0	0	■	1	0	0	6
2	7	1	8	2	8	1	8	2	8